Circular Functions

1 (a) $70^\circ = \frac{7\cancel{0}}{18\cancel{0}} \times \pi = \frac{7\pi}{18}^c$

(b) $300^\circ = \frac{30\cancel{0}}{18\cancel{0}} \times \pi = \frac{5\pi}{3}^c$

(c) $330^\circ = \frac{33\cancel{0}}{18\cancel{0}} \times \pi = \frac{11\pi}{6}$

(d) $18^\circ = \frac{18}{180} \times \pi = \frac{\pi}{10}$

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(2) (a) $\frac{3\pi}{5}^c = \frac{\frac{3\cancel{\pi}}{5}}{\cancel{\pi}} \times 180^\circ = 108^\circ$

(b) $\frac{11\pi}{8}^c = \frac{\frac{11\cancel{\pi}}{8}}{\cancel{\pi}} \times 180 = 247.5^\circ$

(c) $\frac{\pi}{18}^c = \frac{\frac{\pi}{18}}{\pi} \times 180 = 10^\circ$

(d) $2^c = \frac{2}{\pi} \times 180 = \frac{360}{\pi}^\circ \simeq 114.6^\circ$

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(3) (a) $\sin 0 = 0$

(b) $\cos\frac{\pi}{2} = 0$

(c) $\tan 360 = \frac{\sin 360}{\cos 360} = \frac{0}{1} = 0$

(d) $\sin\frac{3\pi}{2} = -1$

(e) $\cos 2\pi = 1$

(f) $\tan 330^\circ = -\tan 30 = -\frac{1}{\sqrt{3}}$

(g) $\cos 3\pi = -1$

(h) $\sin 750 = \sin 30 = \frac{1}{2}$

(i) $\tan\frac{7\pi}{4} = -\tan\frac{\pi}{4} = -1$

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(4) $\cos\theta = -\frac{3}{5}$

$\sin\theta = \frac{4}{5}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{4/5}{-3/5} = -\frac{4}{3}$

$\cos\phi = -\frac{5}{13}$

$\sin\phi = -\frac{12}{13}$

$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{12}{5}$

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(5)

1	+	+	+
2	−	+	−
3	−	−	+
4	+	−	−

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(6) (a) $\sin(-\theta) = -\sin\theta = -\frac{2}{3}$

(b) $\sin(\theta+\pi) = -\sin\theta = -\frac{2}{3}$

(c) $\sin(2\pi-\theta) = -\sin\theta = -\frac{2}{3}$

(d) $\sin(\theta-\pi) = -\sin\theta = -\frac{2}{3}$

(e) $\sin(\theta-2\pi) = \sin\theta = \frac{2}{3}$

(f) $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta + \frac{4}{9} = 1$

$\cos^2\theta = \frac{5}{9}$

$\cos\theta = \pm\frac{\sqrt{5}}{3}$

(g) $\cos(\theta+\pi)$

$= -\cos\theta$

$= -\pm\frac{\sqrt{5}}{3}$

$= \mp\frac{\sqrt{5}}{3}$

(h) $\tan(\theta+\pi)$

$= \frac{\sin(\theta+\pi)}{\cos(\theta+\pi)}$

$= \frac{-2/3}{\mp\frac{\sqrt{5}}{3}}$

$= \pm\frac{2}{\sqrt{5}}$

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(7) (a) $\cos(-\theta)$

$= \cos\theta$

$= \frac{2}{3}$

(b) $\cos(\theta+\pi)$

$= -\cos\theta$

$= -\frac{2}{3}$

(c) $\cos(2\pi-\theta)$

$= \cos\theta$

$= \frac{2}{3}$

(d) $\cos(\theta-\pi)$

$= -\cos\theta$

$= -\frac{2}{3}$

(e) $\cos(\theta-2\pi)$

$= \cos\theta$

$= \frac{2}{3}$

(f) $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \left(\frac{2}{3}\right)^2 = 1$

$\sin\theta = \pm\frac{\sqrt{5}}{3}$

(g) $\sin(\theta+\pi)$

$= -\sin\theta$

$= \mp\frac{\sqrt{5}}{3}$

(h) $\tan(\theta+\pi)$

$= \dfrac{\sin(\theta+\pi)}{\cos(\theta+\pi)}$

$= \dfrac{\pm\frac{\sqrt{5}}{3}}{-\frac{2}{3}}$

$= \pm\frac{\sqrt{5}}{2}$

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(8) (a) $\tan(-\theta)$

$= -\tan\theta$

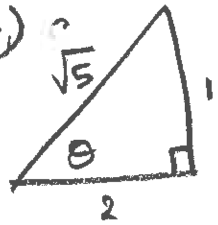
$= -\frac{1}{2}$

(b) $\tan(\theta+\pi)$

$= \tan\theta$

$= \frac{1}{2}$

(c)



$\cos\theta = \pm\frac{2}{\sqrt{5}}$

(d) $\sin\theta = \pm\frac{1}{\sqrt{5}}$

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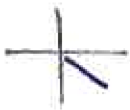
(9) (a) $\sin 30 = \frac{1}{2}$

(b) $\cos\frac{\pi}{3} = \frac{1}{2}$

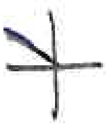
(c) $\tan\left(-\frac{\pi}{6}\right)$



$= -\tan\frac{\pi}{6}$

$= -\frac{1}{\sqrt{3}}$

(d) $\sin\left(\frac{2\pi}{3}\right)$



$= \sin\frac{\pi}{3}$

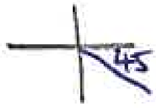
$= \frac{\sqrt{3}}{2}$

(e) $\cos 120^\circ$

$= -\cos 60$

$= -\frac{1}{2}$

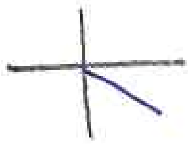
(f) $\sin 315$



$= -\sin 45$

$= -\frac{\sqrt{2}}{2}$ (or $-\frac{1}{\sqrt{2}}$)

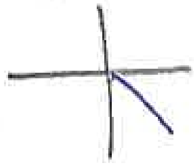
(g) $\tan\frac{11\pi}{6}$



$= -\tan\frac{\pi}{6}$

$= -\frac{1}{\sqrt{3}}$

(h) $\cos\left(\frac{5\pi}{3}\right)$



$= \cos\frac{\pi}{3}$

$= \frac{1}{2}$

(i) $\sin\left(\frac{3\pi}{4}\right)$

$= \sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}$

(j) $\cos\left(-\frac{7\pi}{6}\right)$

$= -\cos\frac{\pi}{6}$

$= -\frac{\sqrt{3}}{2}$

(k) $\tan\left(\frac{8\pi}{3}\right)$

$= -\tan\left(\frac{\pi}{3}\right)$

$= -\sqrt{3}$

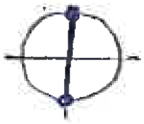
(l) $\cos\left(\frac{25\pi}{3}\right)$

$\frac{25}{3} = 8\frac{1}{3}$

$= \cos\left(\frac{\pi}{3}\right)$

$= \frac{1}{2}$

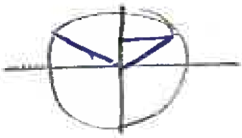
(10) (a) $\cos\theta = 0, \quad 0 \le \theta \le 2\pi$



$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

P.A $= \frac{\pi}{2}$

(b) $\sin\theta = \frac{1}{2}, \quad \theta \in [0, 2\pi]$



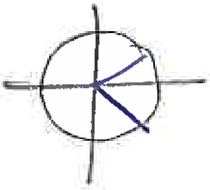
$\theta = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$= \frac{\pi}{6}, \frac{5\pi}{6}$

P.A $= \frac{\pi}{6}$

(c) $2\cos\theta - \sqrt{3} = 0, \quad 0 \le \theta \le 2\pi$

$\cos\theta = \frac{\sqrt{3}}{2}$

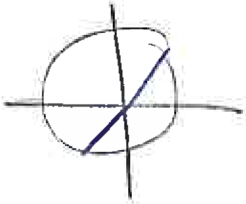


$\theta = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$= \frac{\pi}{6}, \frac{11\pi}{6}$

P.A $= \frac{\pi}{6}$

(d) $\tan\theta = 1, \quad 0 \le \theta \le 4\pi$



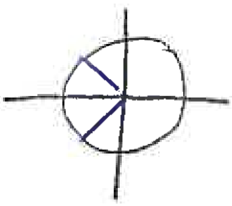
$\theta = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$= \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

P.A $= \frac{\pi}{4}$

Note: In this example 2π is added to the first two solutions $\frac{\pi}{4}, \frac{5\pi}{4}$ since the domain was $0 \le \theta \le 4\pi$. ie We can go twice around the unit circle.

(e) $\cos\theta = -\frac{\sqrt{3}}{2}, \quad -\pi \le \theta \le \pi$

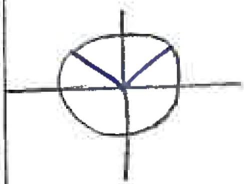


$\theta = \frac{5\pi}{6}, -\frac{5\pi}{6}$

P.A $= \frac{\pi}{6}$

(f) $\sin(2\theta) = \frac{1}{2}, \quad \theta \in [0, 2\pi]$

$2\theta \in [0, 4\pi]$ — $\frac{24\pi}{6}$



$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

P.A $= \frac{\pi}{6}$

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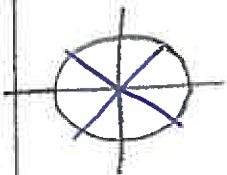
(11) (a) $\tan^2\theta = 1$

$\therefore \tan\theta = \pm 1$



$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

P.A $= \frac{\pi}{4}$

(b) $2\cos^2\theta - 3\cos\theta - 2 = 0$

Let $m = \cos\theta$

$2m^2 - 3m - 2 = 0$

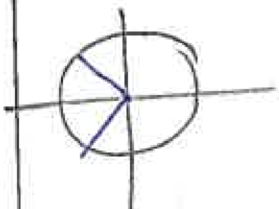
$$\begin{matrix} 2m & & 1 \\ & \times & \\ m & & -2 \end{matrix}$$

$(2m+1)(m-2) = 0$

$m = -\frac{1}{2}, \quad m = 2$

$\therefore \cos\theta = -\frac{1}{2}, \quad \cos\theta = 2$

No solution



$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

P.A $= \frac{\pi}{3}$

(c) $2\sin^2\theta + \sin\theta - 1 = 0$

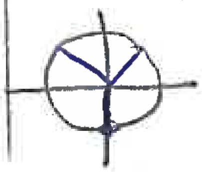
Let $m = \sin\theta$

$2m^2 + m - 1 = 0$

$(2m - 1)(m + 1) = 0$

$m = \frac{1}{2}, -1$

$\therefore \sin\theta = \frac{1}{2}, \quad \sin\theta = -1$



$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

(d) $\sin^2\theta = \sin\theta\cos\theta$

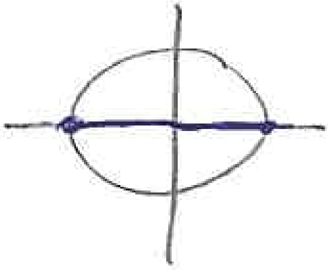
$\sin^2\theta - \sin\theta\cos\theta = 0$

$\sin\theta(\sin\theta - \cos\theta) = 0$

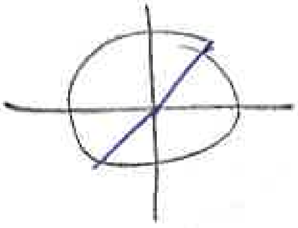
$\sin\theta = 0$ or $\sin\theta - \cos\theta = 0$

$\sin\theta = \cos\theta$

$\therefore \tan\theta = 1$



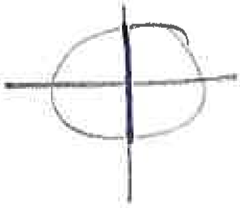
$\theta = 0, \pi, 2\pi$



$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$\theta = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

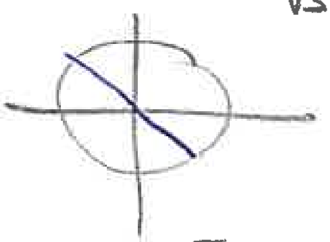
(12) (a) $\cos x = 0$



P.A $= \frac{\pi}{2}$

$x = \frac{\pi}{2} + 2\pi k, \frac{3\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}$

(b) $\tan x = -\frac{1}{\sqrt{3}}$



P.A $= \frac{\pi}{6}$

$x = \frac{5\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k, \quad k \in \mathbb{Z}$

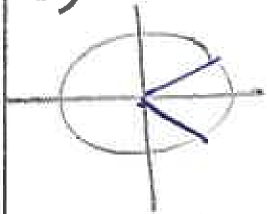
(c) $\sin 2x = -\frac{1}{2}$

P.A $= \frac{\pi}{6}$

$2x = \frac{7\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k, \quad k \in \mathbb{Z}$

$x = \frac{7\pi}{12} + \pi k, \frac{11\pi}{12} + \pi k, \quad k \in \mathbb{Z}$

(d) $\cos 2x = \frac{\sqrt{3}}{2}$

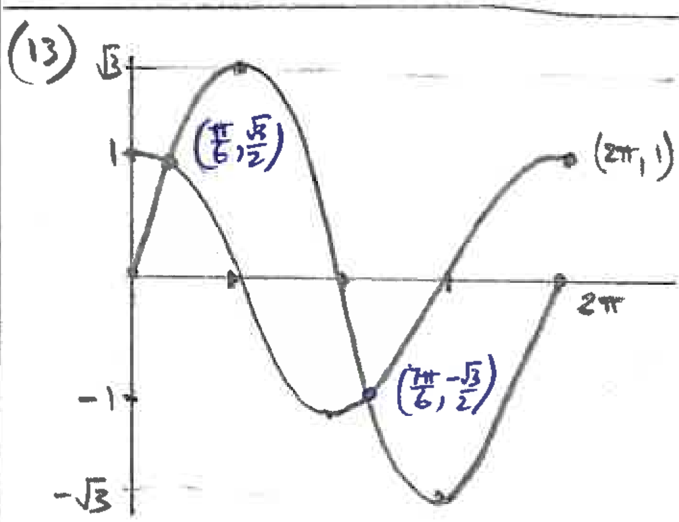


P.A $= \frac{\pi}{6}$

$2x = \frac{\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k$

$x = \frac{\pi}{12} + \pi k, \frac{11\pi}{12} + \pi k, \quad k \in \mathbb{Z}$

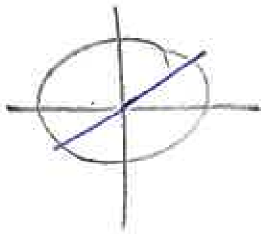
(13)



Let $\cos x = \sqrt{3}\sin x$

$\therefore \frac{1}{\sqrt{3}} = \frac{\sin x}{\cos x}$

$\therefore \tan x = \frac{1}{\sqrt{3}}$



$x = \frac{\pi}{6}, \frac{7\pi}{6}$

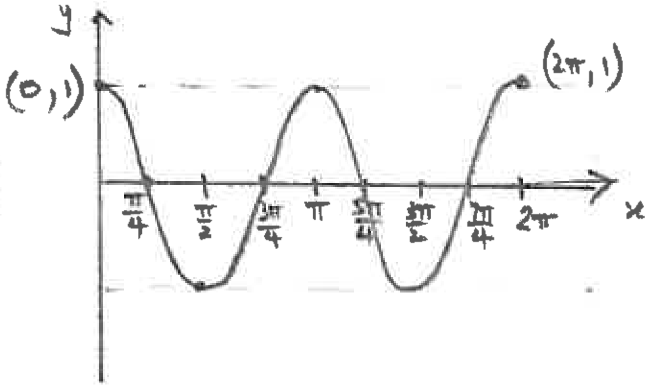
$y = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

P.A $= \frac{\pi}{6}$

$y = \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

(14) (a) $y = \cos 2x, \quad x \in [0, 2\pi]$

$T = \frac{2\pi}{2} = \pi$



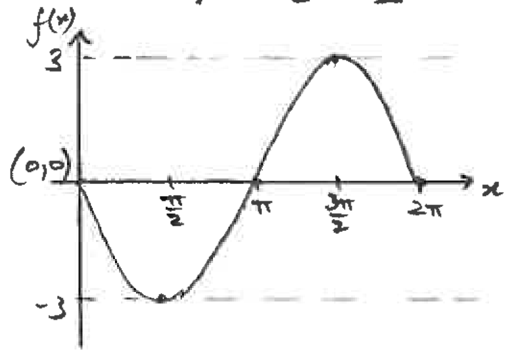
14 (b) $f(x) = -3\sin x$, $x \in [0, 2\pi]$

$T = 2\pi$

Avg $= 0$

Max $= 3$

Min $= -3$



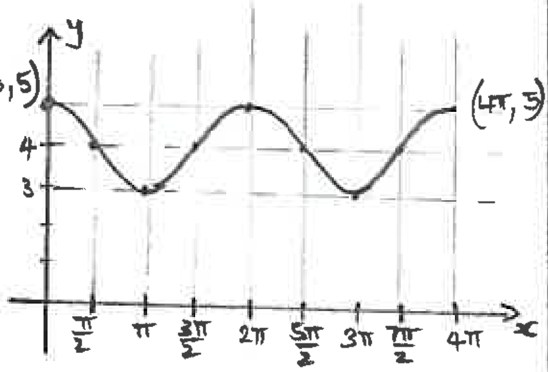
(c) $y = 4 + \cos x$, $x \in [0, 4\pi]$

$T = 2\pi$

Av $= 4$

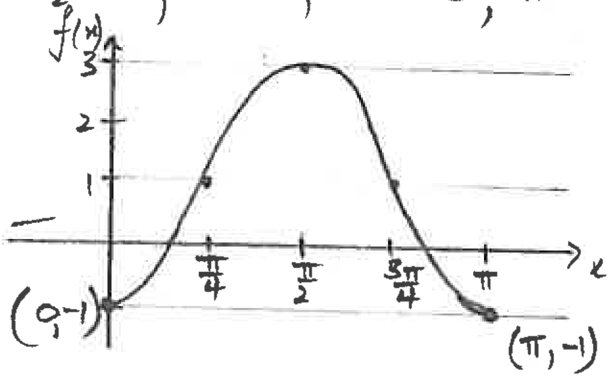
Max $= 5$

Min $= 3$



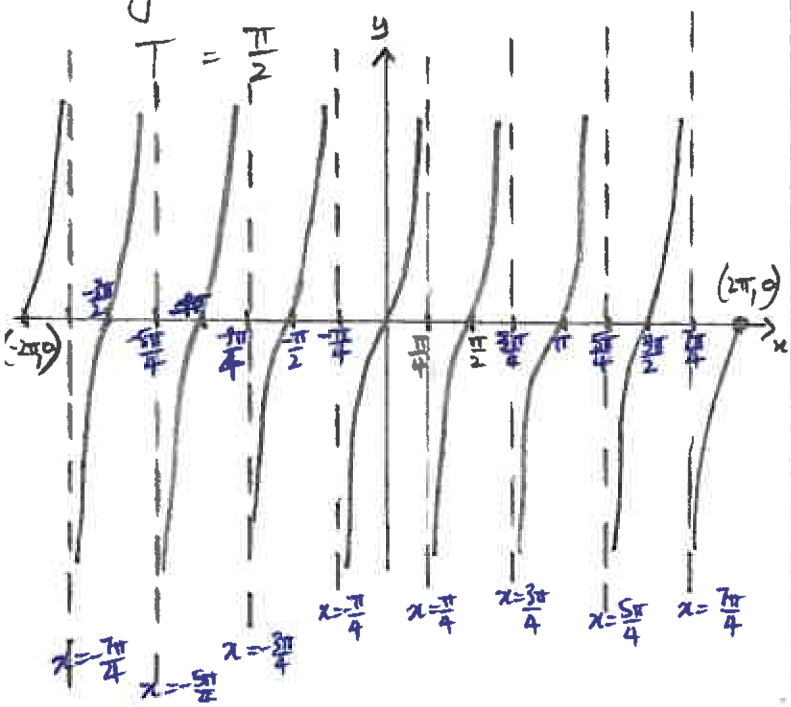
(d) $f(x) = 1 - 2\cos 2x$, $x \in [0, \pi]$

$T = \frac{2\pi}{2} = \pi$, Av $= 1$, Max $= 3$, Min $= -1$



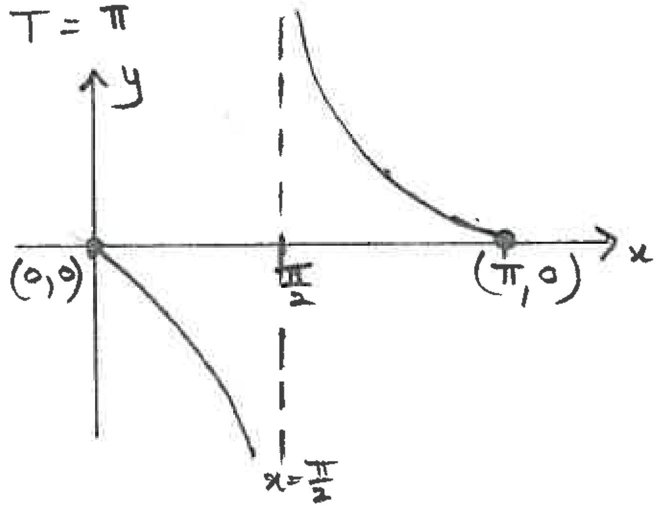
(e) $y = \tan 2x$, $x \in [-2\pi, 2\pi]$

$T = \frac{\pi}{2}$



(f) $y = -\tan x$, $0 \leq x \leq \pi$

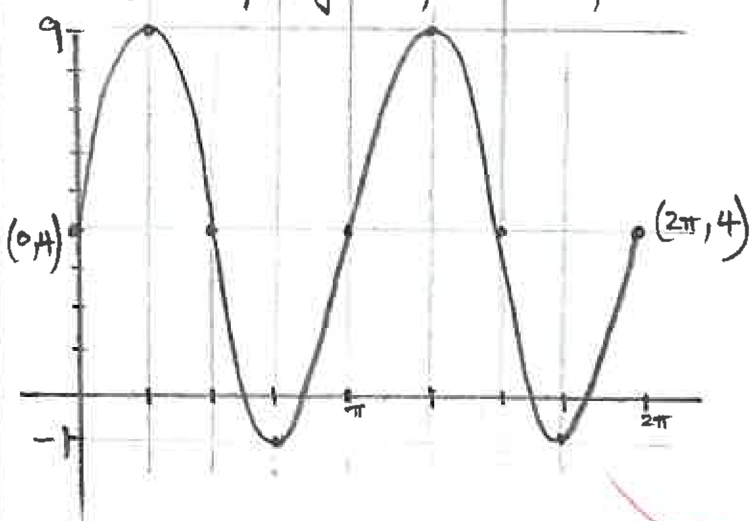
$T = \pi$



* Make sure you always label end points as coordinates and equations of asymptotes.

(15) $f : [0, 2\pi] \to \mathbb{R}$, $f(x) = 4 + 5\sin(2x)$

$T = \frac{2\pi}{2} = \pi$, Avg $= 4$, Max $= 9$, Min $= -1$



(a) $k < -1$ or $k > 9$

(b) $k = -1$ or $k = 9$

(c) $4 < k < 9$ or $-1 < k < 4$

(d) $k = 4$

16 (a) $a = -2$, $b = 3$

$\frac{2\pi}{n} = 2\pi \quad \therefore n = 1$

16 (b) $a = 3$, $b = 2$, $n = 1$

16 (c) $a = 2$, $b = -2$, $n = 1$

16(d) $a=-1,\ b=-2,\ n=1$

(e) $a=-1,\ b=2.$

$\frac{2\pi}{n}=\pi \quad \therefore \ n=2$

(f) $a=-1,\ b=3$

$\frac{2\pi}{n}=\frac{2\pi}{3}$

$\therefore n=3$

(g) $a=-2,\ b=-3$

$\frac{2\pi}{n}=4$

$\therefore \ n=\frac{2\pi}{4}=\frac{\pi}{2}$

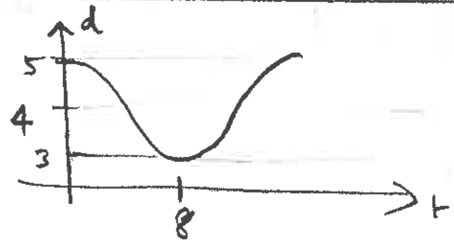
(h) $a=-1,\ b=2$

$\frac{2\pi}{n}=2\times 4=8$

$n=\frac{2\pi}{8}$

$n=\frac{\pi}{4}$

(17)



(a) $d(t)=a+b\cos(nt)$

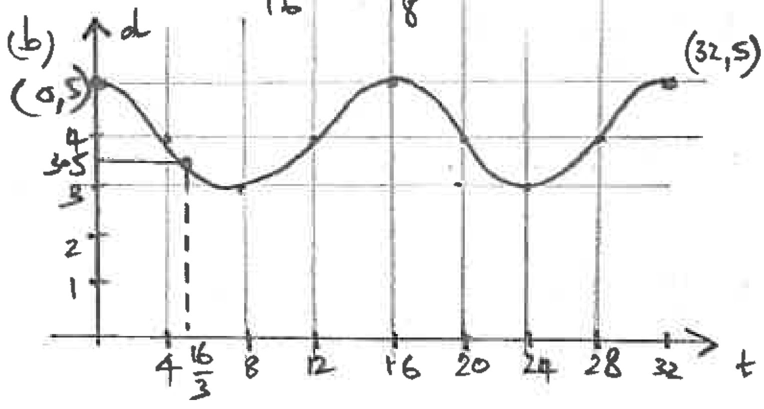
$a=\frac{5+3}{2}=4$

$b=1$

$\frac{2\pi}{n}=16$

$\therefore \ n=\frac{2\pi}{16}=\frac{\pi}{8}$

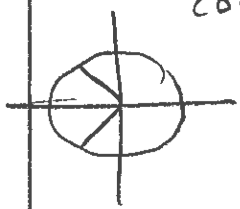
(b)



(c) Let $d=3.5$

$\therefore \ 4+\cos\left(\frac{\pi}{8}t\right)=3.5$

$\cos\left(\frac{\pi t}{8}\right)=-\frac{1}{2}$



$\frac{\pi t}{8}=\frac{2\pi}{3},\ \frac{4\pi}{3}$

P.A $=\frac{\pi}{3}$ $\quad t=\frac{16}{3},\ \frac{32}{3}$

$\frac{16}{3}$ hours after high tide.

(d) $\frac{16/3}{8}=\frac{2}{3}$

$2/3$'s of the time.

(e) Let $d=3$

$4+\cos\left(\frac{\pi t}{8}\right)=3$

$\cos\frac{\pi t}{8}=-1$

$t=8+16k,\ k\in\mathbb{N}$

(f) Let $t=1.5=\frac{3}{2}$

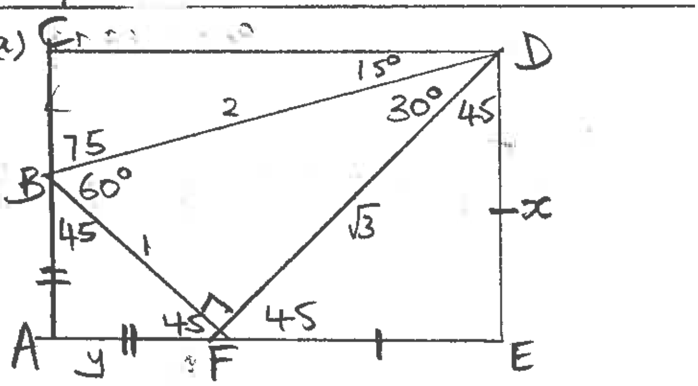
$\therefore d=4+\cos\left(\frac{\pi\times 3/2}{8}\right)$

$=4+\cos\left(\frac{3\pi}{16}\right)$

$=4.83$

No more than 4.83 m up the pylon

18 (a)



$\angle DBF=60°$, $\angle ABF=\angle AFB=45°$

$\angle CBD=75°$, $\angle CDB=15°$,

$\angle EDF=\angle EFD=45°$

18 (b) $x^2 + x^2 = (\sqrt{3})^2$

$2x^2 = 3$

$x^2 = \frac{3}{2}$

$x = \sqrt{\frac{3}{2}}$

(c) $y^2 + y^2 = 1^2$

$2y^2 = 1$

$y^2 = \frac{1}{2}$

$y = \frac{1}{\sqrt{2}}$

(d) $BC = x - y$

$= \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3} - 1}{\sqrt{2}}$

$CD = x + y$

$= \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3} + 1}{\sqrt{2}}$

(e) $\cos 15^\circ = \frac{CD}{2}$

$= \frac{\frac{\sqrt{3}+1}{\sqrt{2}}}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\sin 15^\circ = \frac{CB}{2}$

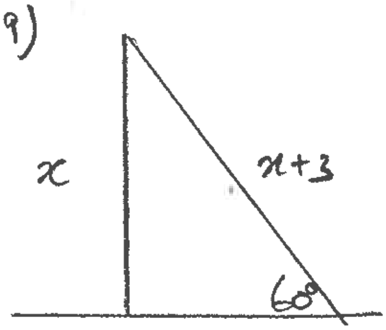
$= \frac{\frac{\sqrt{3}-1}{\sqrt{2}}}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$

(19)



$\sin 60 = \frac{x}{x+3}$

$\frac{\sqrt{3}}{2} = \frac{x}{x+3}$

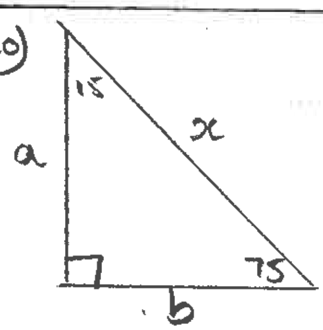
$\sqrt{3}x + 3\sqrt{3} = 2x$

$3\sqrt{3} = 2x - \sqrt{3}x$

$x(2 - \sqrt{3}) = 3\sqrt{3}$

$x = \frac{3\sqrt{3}}{2 - \sqrt{3}}$

(20)



$\cos 15 = \frac{a}{x}$

$\therefore a = x \cos 15$

$= \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) x$

$\sin 15 = \frac{b}{x}$

$b = x \sin 15$

$= \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) x$

$\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) x + \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) x + x = 1$

$x \left[\frac{\sqrt{3}+1 + \sqrt{3}-1 + 2\sqrt{2}}{2\sqrt{2}}\right] = 1$

$x \left[\frac{2\sqrt{3} + 2\sqrt{2}}{2\sqrt{2}}\right] = 1$

$x \left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}}\right) = 1$

$x = \frac{\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

$$\text{Area} = \frac{1}{2}ab$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)$$

$$= \cancel{\frac{1}{2}}\left(\frac{3-1}{8}\right)\left(\frac{2}{3+2\sqrt{6}+2}\right)$$

$$= \frac{1}{4(5+2\sqrt{6})}$$

(21) $y = \sin\left(\frac{\pi}{x}\right)$

x int : $y = 0$

$\sin\left(\frac{\pi}{x}\right) = 0$

$\frac{\pi}{x} = 0 + 2\pi k, \ \pi + 2\pi k, \ k \in \mathbb{Z}$

$\frac{\pi}{x} = 2\pi k, \ \pi + 2\pi k$

$\frac{1}{x} = 2k, \ 1 + 2k$

$x = \frac{1}{2k}, \ \frac{1}{1+2k}$

$k=0, \ x = 1$

$k=1, \ x = \frac{1}{2}, \frac{1}{3}$

$k=-1, \ x = -\frac{1}{2}, -1$

$k=2, \ x = \frac{1}{4}, \frac{1}{5}$

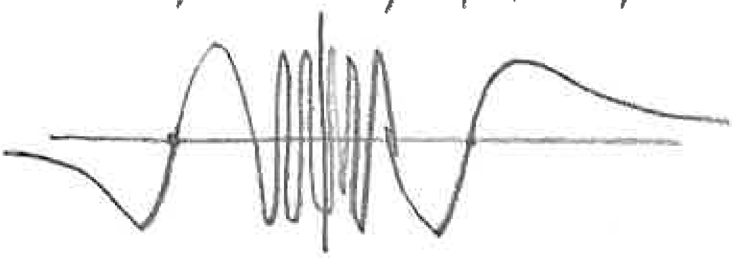
$k=-2, \ x = -\frac{1}{4}, -\frac{1}{3}$

$k=3, \ x = \frac{1}{6}, \frac{1}{7}$

etc

X int are all fractions of the form $\frac{1}{n}$, $n \in \mathbb{Z}\setminus\{0\}$

ie $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{1}{5}, \ldots$



(22)