T2MATHS IS A PREMIUM COACHING SCHOOL DELIVERING THE HIGHEST QUALITY TEACHING FOR SECONDARY SCHOOL MATHEMATICS. OUR INSTRUCTORS DRAW FROM EXTENSIVE TEACHING EXPERIENCE IN SUBJECTS INCLUDING: VCE MATHEMATICAL METHODS, VCE SPECIALIST MATHEMATICS, IB MATHEMATICS AND UNIVERSITY ENHANCEMENT MATHEMATICS. T2MATHS INSTRUCTORS ARE MATHEMATICALLY ACCOMPLISHED, ENGAGING, AND CURRENTLY TEACH AT TOP INDEPENDENT SCHOOLS, SUPPORTING STUDENTS TO ACHIEVE OUTSTANDING RESULTS.

SAMPLE VCE METHODS EXAM 1 & EXAM 2

EXAM 1 (Total 40 marks)

Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working must be shown.

Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Question 1 (5 marks)

a. Let $y = \sqrt{x^2 - 1}$. Find $\frac{dy}{dx}$. (2 marks)

b. Let $f(x) = \frac{x}{1 + \sin x}$. Find $f'(\pi)$. (3 marks)

Question 2 (6 marks)

a. Let $f'(x) = \sqrt{x} - x^2$.

Given that $f(3) = 2$, find $f(x)$. (3 marks)

b. Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (2\cos(2x) + \sin(3x))\,dx$. (3 marks)

Question 3 (5 marks)

Let $y = \log(\cos(x))$.

a. Find $\dfrac{dy}{dx}$. (2 marks)

b. Hence, evaluate $\int_0^{\frac{\pi}{3}} 5\tan(x)\,dx$. (3 marks)

Question 4 (3 marks)

Let Z be the standard normal random variable, such that $Z \sim \mathrm{N}(0,1)$.

Suppose that $\Pr(Z \geq a) = b$ where $a > 0$ and $0 < b < 1$.

a. Find $\Pr(-a \leq Z \leq a)$ in terms of b. (1 marks)

b. Find $\Pr(Z \geq -a \mid Z \leq a)$ in terms of b. (2 marks)

Question 5 (4 marks)

A fair coin has heads on one side and tails on the other. The coin is tossed five times.

a. Find the probability that heads is obtained at least once. (2 marks)

b. Find the probability that heads is obtained at least once given that tails is obtained at least once. (2 marks)

Question 6 (5 marks)

Suppose that $b > a$. The random variable X has the discrete probability distribution shown below.

x	1	a	b
$\Pr(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

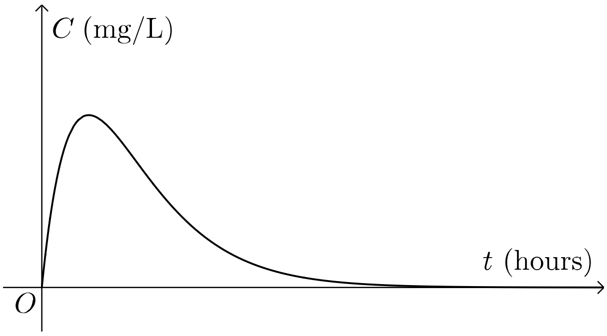
a. It is known that $\text{E}(X) = 1$. Find the value of $a + b$. (1 marks)

b. It is also known that $\text{SD}(X) = 1$. Find the value of $a^2 + b^2$. (2 marks)

c. Find the values of a and b. (2 marks)

Question 7 (4 marks)

The concentration C mg/L of a drug in a patient's blood t hours after begin administered is given by $C(t) = 10te^{-t}$. The graph of C against t is shown below.



a. Find $C'(t)$. (2 marks)

b. Find the time t for which the concentration is a maximum. (2 marks)

Question 8 (3 marks)

Consider the binomially distributed discrete random viable $X \sim \mathrm{B}(5, p)$, where $p \in [0, 1]$.

Given that $\Pr(X = 5 \,|\, X \geq 4) = \frac{1}{3}$, find the value of p.

Question 9 (5 marks)

Let a be some real number. Consider the function $f : [-1, 1] \to \mathbb{R}, f(x) = ax^2 - 4x + 1$.

a. For which values of a will the function f have an inverse? (2 marks)

Let k be some real number. Consider the function $g : \mathbb{R} \to \mathbb{R}, f(x) = 2x^3 - kx^2 + 2x + 1$.

b. For which values of k is the function g one to one? (3 marks)

EXAM 2, SECTION 1. (Total 20 marks)

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Question 1

If the range of $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 + 2kx + 3k^2$ is $[18, \infty)$ then

A. $k = \pm 1$

B. $k = \pm 2$

C. $k = \pm 3$

D. $k = \pm 4$

E. $k = \pm 5$

Question 2

The inverse of the function $f : [1, 3) \to \mathbb{R}, f(x) = e^{x-1} + 1$ is

A. $f^{-1} : [1, 3) \to \mathbb{R}, f^{-1}(x) = 1 + \log_e(x + 1)$

B. $f^{-1} : (2, e^2 + 1) \to \mathbb{R}, f^{-1}(x) = -1 + \log_e(x - 1)$

C. $f^{-1} : [2, e^2 + 1) \to \mathbb{R}, f^{-1}(x) = 1 + \log_e(x - 1)$

D. $f^{-1} : (2, e^2 + 1) \to \mathbb{R}, f^{-1}(x) = 1 + \log_e(x + 1)$

E. $f^{-1} : (2, e^2 + 1] \to \mathbb{R}, f^{-1}(x) = 1 + \log_e(x - 1)$

Question 3

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the rule

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

This transformation is applied to the function $f : \mathbb{R} \to \mathbb{R}, f(x) = 4\cos(2x) + 2$.
The range of the image is

A. $[-4, 2]$

B. $[-9, 15]$

C. $[-9, 0]$

D. $[0, 15]$

E. $[-2, 4]$

Question 4

Let $f : \mathbb{R} \to \mathbb{R}$ be some function. A transformation that maps the graph of $y = f(2x + 6)$ onto the graph of $y = f(2x + 4)$ is

A. a translation by 1 unit to the right.

B. a translation by 1 unit to the left.

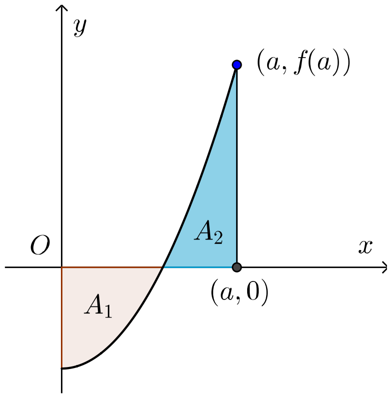
C. a translation by 2 units to the right.

D. a translation by 2 units to the left.

E. a translation by 4 units to the right.

Question 5

For $a > 0$ consider the function $f : [0, a] \to \mathbb{R}, f(x) = 3x^2 - 3$.



The shaded region below the x-axis has area A_1.
The shaded region above the x-axis has area A_2.
If $A_1 = A_2$ then a is equal to

A. 1

B. 2

C. $\sqrt{2}$

D. 3

E. $\sqrt{3}$

Question 6

Let $k > 0$ and consider the function $f : [0, k] \to \mathbb{R}, f(x) = \sin x$. If the area bounded by the graph of f and the x-axis is 11, then k is equal to

A. $\frac{11\pi}{2}$

B. $\frac{13\pi}{2}$

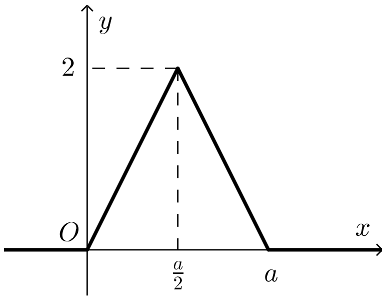
C. 5π

D. 6π

E. 11π

Question 7

The graph of the probability density function of a continuous random variable, X, is shown below.



If $a > 0$, then $\mathrm{E}(X)$ is equal to

A. $\dfrac{1}{2}$

B. 2

C. $\sqrt{2}$

D. 3

E. $\sqrt{3}$

Question 8

There are three red and five blue marbles in a bag. Sally selects three balls without replacement. The probably that she selects at least one ball of each colour is

A. $\dfrac{11}{56}$

B. $\dfrac{45}{56}$

C. $\dfrac{55}{56}$

D. $\dfrac{23}{28}$

E. $\dfrac{5}{28}$

Question 9

There are three girls and an unknown number of boys in a group. If two children are selected without replacement then the probability that both children are boys is $\frac{7}{15}$.
The number of boys in the group is

A. 5

B. 6

C. 7

D. 8

E. 9

Question 10

Consider the following discrete probability distribution for the random variable X.

x	0	1	2	3
$\Pr(X = x)$	p	p	$3p$	$\frac{1}{6}$

The mean of this distribution is equal to

A. $\frac{1}{6}$

B. $\frac{5}{3}$

C. $\frac{5}{6}$

D. $\frac{1}{2}$

E. $\frac{5}{2}$

Question 11

For which values of k will the function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + kx^2 + 3x + 1$ have an inverse?

A. $-3 \leq k \leq 3$

B. $-4 \leq k \leq 4$

C. $k \geq 3$

D. $k \geq 4$

E. $k \leq -4$

Question 12

If the graph of $y = (x + a)e^{2x}$ has a turning point at $x = -1$, then a is equal to

A. $\frac{1}{2}$

B. 1

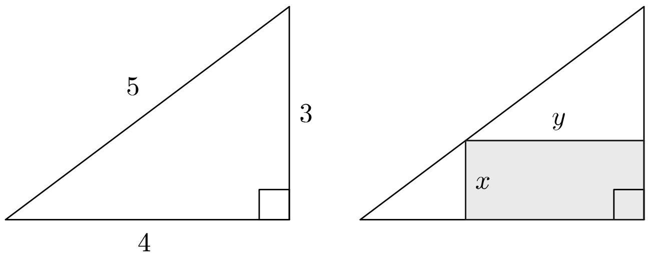
C. 2

D. -1

E. -2

Question 13

A rectangle is drawn inside the right-angled triangle below so that each vertex lies on an edge of the triangle.



The maximum area of the triangle is

A. $\frac{3}{2}$

B. $\frac{2}{3}$

C. 2

D. $\frac{1}{2}$

E. 3

Question 14

Suppose X is normally distributed with mean $\mu = 3$. If $\Pr(X > 4) = \frac{1}{3}$ then $\Pr(X < 4 \mid X > 2)$ is equal to

A. $\frac{3}{4}$

B. $\frac{2}{3}$

C. $\frac{1}{2}$

D. $\frac{1}{3}$

E. $\frac{1}{4}$

Question 15

If $\int_0^\pi f(x)\,dx = 2$ and $\int_0^\pi \left(2f(x) + a\cos(\frac{x}{2})\right)\,dx = 10$ then a is equal to

A. 1

B. 2

C. 3

D. 4

E. 5

Question 16

If the tangent to the graph of $y = a\log_e(2x-1)+1$ at $x=1$ goes through the origin, then a is equal to

A. $\frac{1}{2}$

B. 1

C. 2

D. -1

E. -2

Question 17

There are four million beads in a container, and three million of these are black. For each sample of 15 beads, $\hat{P}$ is the random variable for the sample proportion of black beads. (Do not use a normal approximation.)

$\Pr(\hat{P} \geq \frac{2}{3})$ is closest to:

A. 0.8516

B. 0.6865

C. 0.5332

D. 0.3135

E. 0.1651

Question 18

If the graph of $f(x) = a\sqrt{x} + \frac{b}{\sqrt{x}}$ has a turning point at coordinates $(1, 1)$ then

A. $a = 1, b = -1$

B. $a = 1, b = 1$

C. $a = 1, b = \frac{1}{2}$

D. $a = \frac{1}{2}, b = 1$

E. $a = \frac{1}{2}, b = \frac{1}{2}$

Question 19

Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2 - 2x - 8$ and $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x^2 + c$.
For which values of c will the graph of $y = f(g(x))$ have no x-intercepts?

A. $c > -8$

B. $c < 8$

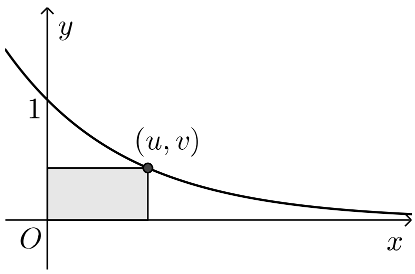
C. $c < 2$

D. $c > 4$

E. $c > 1$

Question 20

In the diagram below, a rectangle is drawn in the first quadrant. Two of its sides lie along the coordinate axes. One vertex (u, v) lies on the graph of $f(x) = e^{-x}$.



The maximum area of the rectangle is

A. 1

B. e

C. 2

D. e^2

E. $\frac{1}{e}$

EXAM 2, SECTION 2 (Total 60 marks)

Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working must be shown.

Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Question 1 (16 marks)

When a client calls a bank the *wait time* in hours to speak to an operator is a continuous random variable X whose probability density function is

$$f(x) = \begin{cases} \frac{a}{x+1}, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

a. Show that $a = \frac{1}{\log_e(2)}$. (2 marks)

b. Find the mean wait time taken for a customer to speak to an operator, correct to two decimal places. (1 mark)

c. Find the median wait time taken for a customer to speak to an operator, correct to two decimal places. (2 marks)

d. For 20% of customers, the wait time is more than k minutes. Find the value of k, correct to two decimal places. (2 marks)

e. If a customer calls the bank, find the probability that the wait time is less than thirty minutes, correct to two decimal places. (2 marks)

f. If a customer calls the bank, find the probability that the wait time is more than fifteen minutes given that it is less than thirty minutes, correct to two decimal places. (2 marks)

g. A customer calls the bank once a day for seven days. Find the probability that the wait time is less than thirty minutes on four days out of these seven days. Answer correct to two decimal places. (2 marks)

The bank would like know how satisfied the customers are with its telephone service. They sample 2000 customers. Of these customers, 764 say that they are satisfied.

h. Find the sample proportion of satisfied customers. (1 mark)

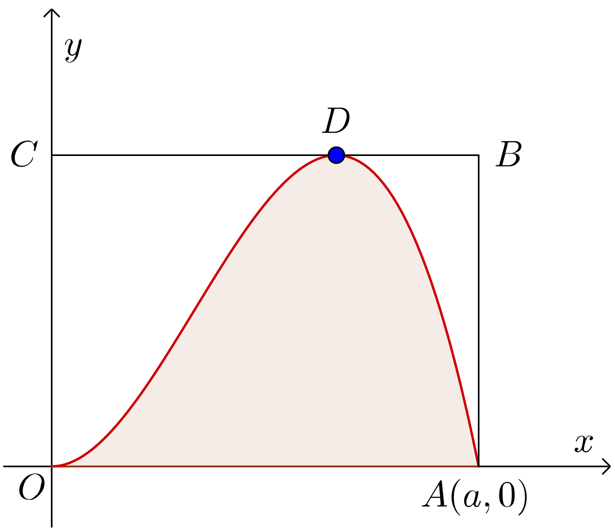
i. Find the 99% confidence interval for the proportion of satisfied customers, accurate to two decimal places (2 marks)

Question 2 (12 marks)

Suppose $a > 0$ and let $f : [0, a] \to \mathbb{R}$ be defined by

$$f(x) = \frac{27}{a^4} x^2 (a - x).$$

The graph of f is shown on the axes below.



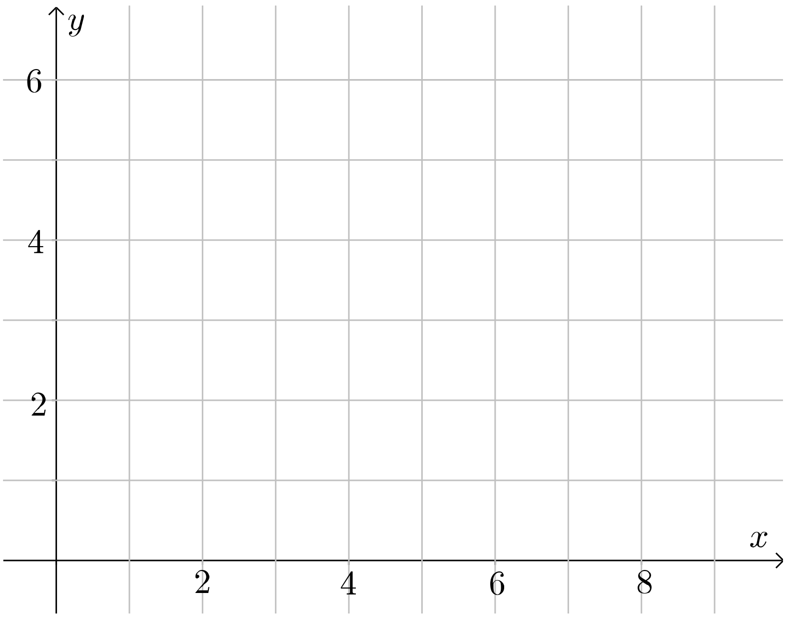
a. Point D is the maximum of the graph of f. Find its coordinates in terms of a. (2 marks)

b. Find the area of the rectangle $OABC$. (1 mark)

c. The area beneath the graph of f has been shaded. Find the fraction of the rectangle's area that is shaded. (2 marks)

Now let $g : [0, 6] \to \mathbb{R}$ be defined by $g(x) = \frac{1}{8}x^2(6 - x)$.

d. Sketch the graph of g on the axes below, labelling end-points and turning points with their coordinates. (3 marks)



e. The gradient of g is a maximum at coordinate (m, n). Find the values of m and n. (3 marks)

f. Define $h(x) = g(x + m) - n$. Show that $h(-x) = -h(x)$. (2 marks)

Question 3 (11 marks)

Bottles of water are produced at a factory by two machines. One is called machine A and the other is called machine B.

Machine A fills bottles whose volumes are normally distributed with a mean volume of 502 ml and standard deviation of 2 ml.

Bottles of water are rejected if their volume is less that 497 ml.

a. If 99% of bottles produced by machine A have a volume less than c, then find the value of c, correct to 2 decimal places. (2 marks)

b. Find the probability that machine A produces a bottle that is rejected, correct to 4 decimal places. (2 marks)

Machine B fills bottles whose volumes are also normally distributed. It is known that 15% of the the bottle have a volume greater than 507 ml and 2% have a volume less than 498 ml.

c. Find the mean and standard deviation of bottles produced by machine B, correct to 4 decimal places. (3 marks)

d. Find the probability that machine B produces a bottle that is rejected, correct to 4 decimal places. (2 marks)

e. 0.7% of the combined output from both machines is rejected. Find the percentage of bottles that are produced by machine A, to the nearest percent. (2 marks)

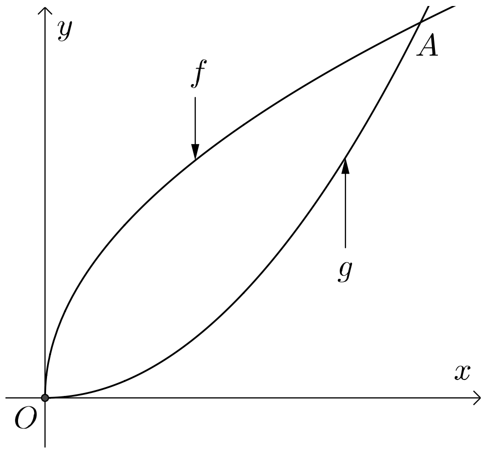
Question 4 (20 marks)

Consider two functions

$$f : [0, \infty) \to \mathbb{R} \text{ where } f(x) = \sqrt{x},$$

$$g : [0, \infty) \to \mathbb{R} \text{ where } g(x) = x^2.$$

The graphs of these two functions are shown below.



a. Write down the coordinates of the point of intersection A. (1 mark)

b. Find the area bounded by the graphs of f and g. (2 mark)

c. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ where

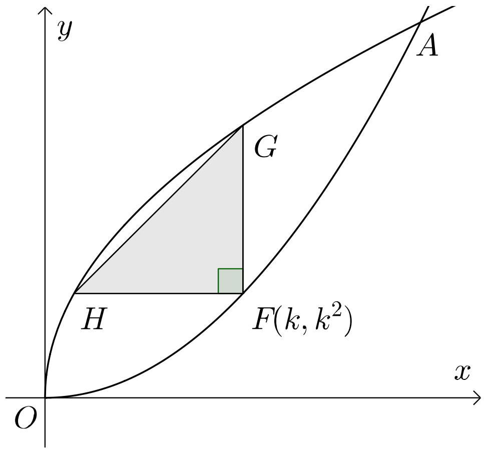
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

If this transformation was applied to both functions f and g, then find the area bounded by the image of these functions. (1 mark)

d. The vertical line $x = \frac{1}{4}$ intersects the graphs of f and g at the points B and C, respectively. Find the distance BC. (2 mark)

e. If $0 \leq k \leq 1$, then the vertical line $x = k$ intersects the graphs of f and g at the points D and E, respectively. Find the exact value of k if the distance DE is a maximum. (3 marks)

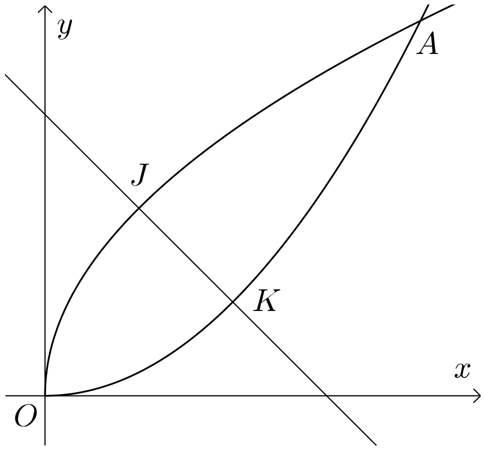
A right-angled triangle FGH is drawn in the region bounded by the two graphs. Two sides are parallel to the coordinate axes. Point F has coordinate (k, k^2), where $0 \leq k \leq 1$.



f. Find the coordinates G and H in terms of k. (2 marks)

g. Let $A(k)$ be the area of triangle FGH. Find the value of k that maximises the area of the triangle, accurate to two decimal places. (3 marks)

Suppose $0 \leq c \leq 2$. The straight line $x + y = c$ intersects the graph of f and g at the points J and K, respectively.



h. If $c = \frac{3}{4}$, find the area of the region whose boundary contains points O, J and K (3 marks)

i. Find the value of c if the straight line divides the region bounded by the graphs of f and g into two regions of equal area. Give your answer accurate to 2 decimal places. (3 marks)