1 Exercises

Ex1 1. Find the equation of the line perpendicular to $y = 3x + 2$ that goes through the points $A(5, 1)$. Find the shortest distance from the point to the given line.

Ex1 2. Find the equation of the line perpendicular to $y = -2x + 1$ that goes through the points $A(3, 3)$. Find the shortest distance from the point to the given line.

Ex2 3. (a) Sketch the graph of $y = x^2 + 4x - 10$ including all significant features.

(b) Hence, find all values of x such that $x^2 + 4x < \cancel{5}$ 10

Ex2 4. (a) Sketch the graph of $y = -x^2 + 2x + 5$ including all significant features.

(b) Hence, find all values of x such that $x^2 \geq 2x + 5$.

Ex3 5. Consider the function $f : [\frac{3}{2}, \infty) \to \mathbb{R}$ where $f(x) = \frac{1}{2x-3} + 1$

(a) Sketch the graph of f

(b) Find the domain and range of f.

(c) Find the rule for the inverse of f.

Ex3 6. Consider the function $f : [-1, \infty) \to \mathbb{R}$ where $f(x) = -\sqrt{x+1} + 1$

(a) Sketch the graph of f

(b) Find the domain and range of f.

(c) Find the rule for the inverse of f.

Ex4 7. Suppose $\Pr(A) = 0.2$, $\Pr(B') = 0.5$ and $\Pr(A \cap B) = 0.1$.

(a) Find:

(i) $\Pr(B)$ (ii) $\Pr(A' \cap B')$ (iii) $\Pr(A' \cup B)$ (iv) $\Pr(B'|A)$.

(b) Are events A and B independent?

Ex4 8. Suppose $\Pr(A') = 0.5$, $\Pr(B) = 0.4$ and $\Pr(A' \cap B) = 0.2$.

(a) Find:

(i) $\Pr(B')$ (ii) $\Pr(A \cap B')$ (iii) $\Pr(A' \cup B)$ (iv) $\Pr(B|A)$.

(b) Are events A and B independent?

Ex5 9. Sketch the graph of $y = x^3 + 4x^2 - 17x - 60$, labelling the coordinates of intercepts and turning points.

Ex5 10. Sketch the graph of $y = x^3 - 2x^2 - 15x + 36$, labelling the coordinates of intercepts and turning points.

Ex6 11. (a) Solve $\sin(\theta) = -\frac{1}{2}$ for $0 \leq \theta \leq 2\pi$.

(b) Solve $\cos(2\theta) = \frac{1}{\sqrt{2}}$ for $0 \leq \theta \leq 2\pi$.

(c) Sketch the graph of $y = 2\sin\left(\frac{x}{2}\right) + 1$ over the interval $0 \leq x \leq 2\pi$

Ex6 12. (a) Solve $2\sin(\theta) = 1$ for $0 \leq \theta \leq 2\pi$.

(b) Solve $\tan(2\theta) = -\sqrt{3}$ for $0 \leq \theta \leq 2\pi$.

(c) Sketch the graph of $y = -2\cos(2x) - 1$ over the interval $0 \leq x \leq 4\pi$

Ex7 13. (a) Solve $9^{2x-1} = 3\sqrt{3}$ for x.

(b) Solve $\log_3(x^2 - 2x - 8) = 3$ for x.

(c) Sketch the graph of $y = 2^{-x} - 8$.

(d) Sketch the graph of $y = \log_{10}(4 - 2x)$.

Ex7 14. (a) Solve $25^{x-2} = \sqrt{5^x}$ for x.

(b) Solve $\log_2(x^2 - 4x + 11) = 4$ for x.

(c) Sketch the graph of $y = -2^{x+1} - 4$.

(d) Sketch the graph of $y = -\log_2(2x - 3)$.

Ex8 15. Find the equation of the tangent to the graph of $y = x^2 + 3x - 5$ at the point $x = 1$.

Ex8 16. Find the equation of the tangent to the graph of $y = 3\sqrt{x} - \frac{3}{x^2} + 2$ at the point $x = 1$.

Ex9 17. Find the area bounded by the graph of $y = x(1 - x)$ and the lines $y = 0$ and $x = 2$.

Ex9 18. Find the area bounded by the graph of $y = x^2 - 1$, the x-axis and the lines $x = 0$ and $x = 2$.

Ex10 19. Consider the graph of $f(x) = 4 - x^2$ over the interval $-2 \leq x \leq 2$. Sketch the graph of this function. A rectangle is drawn so that two of its vertices are on the x-axis and the other two are on the graph of f. Find the area of the rectangle if:

(a) its base has width 3,

(b) its area is as large as possible,

(c) its area is area is exactly one third of the area bound by the graph of f and the x-axis.

Ex10 20. Consider the graph of $f(x) = 1 - x^2$ over the interval $-1 \leq x \leq 1$. Sketch the graph of this function. A rectangle is drawn so that two of its vertices are on the x-axis and the other two are on the graph of f. Find the area of the rectangle if:

(a) its base has width 1,

(b) its area is as large as possible,

(c) its area is area is exactly one quarter of the area bound by the graph of f and the x-axis.

Consolidation

(1) $y = 3x + 2$

$\Rightarrow m_1 = 3$

$\Rightarrow m_2 = -\frac{1}{3}$

$A(5,1)$

$y - y_1 = m(x - x_1)$

$y - 1 = -\frac{1}{3}(x - 5)$

$y = -\frac{1}{3}x + \frac{5}{3} + 1$

$\boxed{y = -\frac{1}{3}x + \frac{8}{3}}$

Point of intersection

$-\frac{1}{3}x + \frac{8}{3} = 3x + 2$

$-x + 8 = 9x + 6$

$10x = 2$

$x = \frac{1}{5}$

$y = 3 \times \frac{1}{5} + 2$

$= \frac{13}{5}$

$\left(\frac{1}{5}, \frac{13}{5}\right) \quad A(5,1)$

$\therefore$ Shortest Distance $= \sqrt{\left(1 - \frac{13}{5}\right)^2 + \left(5 - \frac{1}{5}\right)^2}$

$= \sqrt{\left(\frac{-8}{5}\right)^2 + \left(\frac{24}{5}\right)^2}$

$= \sqrt{\frac{640}{25}}$

$= \boxed{\frac{8\sqrt{10}}{5}}$

(2) $y = -2x + 1$

$\therefore m_1 = -2$

$m_2 = \frac{1}{2}$

$A(3,3)$

$\therefore y - 3 = \frac{1}{2}(x - 3)$

$y = \frac{1}{2}x - \frac{3}{2} + 3$

$\boxed{y = \frac{1}{2}x + \frac{3}{2}}$

Pt of intersection:

$-2x + 1 = \frac{1}{2}x + \frac{3}{2}$

$-4x + 2 = x + 3$

$5x = -1$

$x = -\frac{1}{5}$

$x = -\frac{1}{5}, \; y = -2 \times -\frac{1}{5} + 1 = \frac{2}{5} + 1 = \frac{7}{5}$

$\left(-\frac{1}{5}, \frac{7}{5}\right) \quad A(3,3)$

$D = \sqrt{\left(3 - \frac{7}{5}\right)^2 + \left(3 + \frac{1}{5}\right)^2}$

$= \sqrt{\frac{64}{25} + \frac{256}{25}}$

$= \frac{8\sqrt{5}}{5}$

(3) (a) $y = x^2 + 4x - 10$

$= (x+2)^2 - 4 - 10$

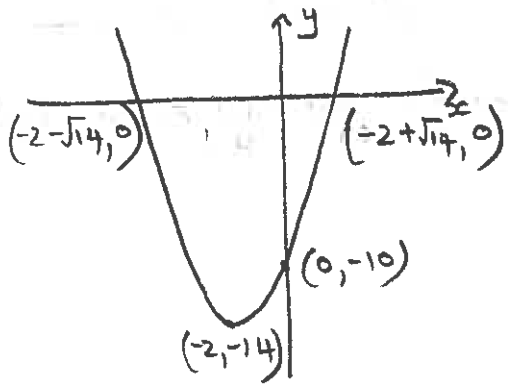
$= (x+2)^2 - 14$

Y int: $x = 0, \; y = -10 \quad (0, -10)$

X int: $(x+2)^2 - 14 = 0$

$x + 2 = \pm\sqrt{14}$

$x = -2 \pm \sqrt{14}$



(b) $x^2 + 4x < 10$

$x^2 + 4x - 10 < 0$

$-2 - \sqrt{14} < x < -2 + \sqrt{14}$

4(a) $y = -x^2 + 2x + 5$

$= -[x^2 - 2x - 5]$

$= -[(x-1)^2 - 1 - 5]$

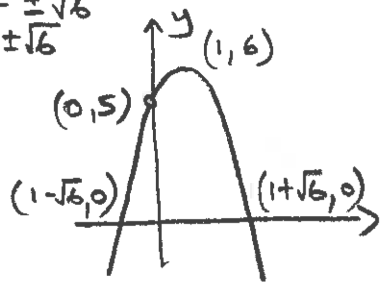
$= -(x-1)^2 + 6$

Y int: $x = 0, \; y = 5 \quad (0,5)$

X int: $-(x-1)^2 + 6 = 0$

$x - 1 = \pm\sqrt{6}$

$x = 1 \pm \sqrt{6}$



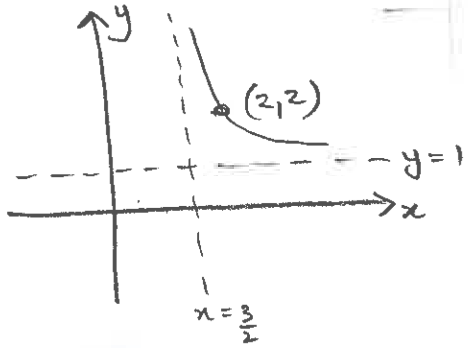
(b) $x^2 \geqslant 2x + 5$

$-x^2 + 2x + 5 \leqslant 0$

$x \leqslant 1 - \sqrt{6}$ or $x \geqslant 1 + \sqrt{6}$

5) $f:\left(\frac{3}{2},\infty\right)\to\mathbb{R},\ f(x)=\frac{1}{2x-3}+1$

(a) $f(x)=\frac{1}{2\left(x-\frac{3}{2}\right)}+1$



$x=2,\ f(2)=\frac{1}{4-3}+1=2 \quad (2,2)$

(b) $\text{dom } f=\left(\frac{3}{2},\infty\right)$

$\text{ran } f=(1,\infty)$

(c) Let $y=\frac{1}{2x-3}+1$

swap y and x

$\therefore x=\frac{1}{2y-3}+1$

$x-1=\frac{1}{2y-3}$

$\frac{1}{x-1}=2y-3$

$2y=\frac{1}{x-1}+3$

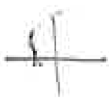
$y=\frac{1}{2(x-1)}+\frac{3}{2}$

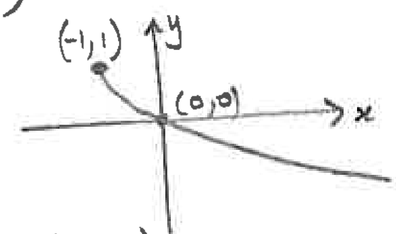
$f^{-1}(x)=\frac{1}{2x-2}+\frac{3}{2}$

6 (a) $f:[-1,\infty)\to\mathbb{R},\ f(x)=-\sqrt{x+1}+1$

Y int: $x=0,\ y=-\sqrt{1}+1=0$

$(0,0)$





(b) $\text{dom } f=[-1,\infty)$

$\text{ran } f=(-\infty,1]$

(c) let $y=-\sqrt{x+1}+1$

swap x and y

$x=-\sqrt{y+1}+1$

$\sqrt{y+1}=1-x$

$y+1=(1-x)^2$

$y=(1-x)^2-1$

$=(x-1)^2-1$

$f^{-1}(x)=(x-1)^2-1$

(7) $\Pr(A)=0.2,\ \Pr(B)=0.5,\ \Pr(A\cap B)=0.1$

	A	A'	
B	0.1	0.4	0.5
B'	0.1	0.4	0.5
	0.2	0.8	1

(a) (i) $\Pr(B)=0.5$

(ii) $\Pr(A'\cap B')=0.4$

(iii) $\Pr(A'\cup B)$

$=\Pr(A')+\Pr(B)-\Pr(A'\cap B)$

$=0.8+0.5-0.4$

$=0.9$

(iv) $\Pr(B'|A)=\frac{\Pr(B'\cap A)}{\Pr(A)}$

$=\frac{0.1}{0.2}$

$=\frac{1}{2}$

(b) $\Pr(B'|A)=\Pr(B')$

Yes independent.

(8)

	B	B'	
A	0.2	0.3	0.5
A'	0.2	0.3	0.5
	0.4	0.6	1

(a) (i) $\Pr(B')=0.6$ (ii) $\Pr(A\cap B')=0.3$

(iii) $\Pr(A'\cup B)=\Pr(A')+\Pr(B)-\Pr(A'\cap B)$

$=0.5+0.4-0.2$

$=0.7$

(iv) $\Pr(B|A)=\frac{\Pr(B\cap A)}{\Pr(A)}=\frac{0.2}{0.5}=\frac{2}{5}=0.4$

(b) $\Pr(B|A)=\Pr(B)$

$\therefore$ Events independent.

(9) $y = x^3 + 4x^2 - 17x - 60$

Y int: $x = 0,\ y = -60 \quad (0, -60)$

$x = -2,\ y = (-2)^3 + 4(-2)^2 - 17(-2) - 60 = -18 \quad \times$

$x = -5,\ y = (-5)^3 + 4(-5)^2 - 17(-5) - 60 = 0$

$\therefore\ x + 5$ is a factor

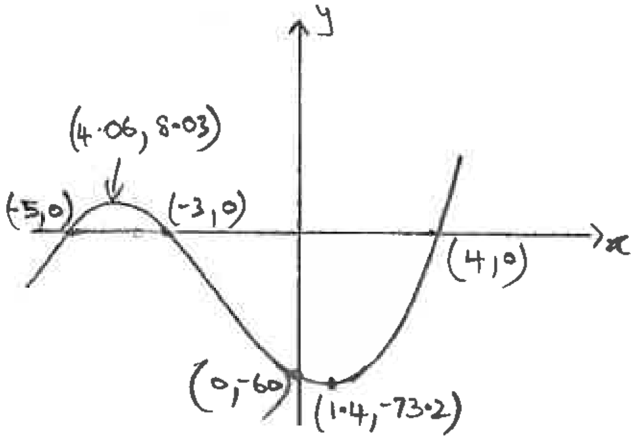
$$\begin{array}{r|l} & x^2 - x - 12 \\ x+5 & x^3 + 4x^2 - 17x - 60 \\ & x^3 + 5x^2 \\ \hline & -x^2 - 17x - 60 \\ & -x^2 - 5x \\ \hline & -12x - 60 \\ & -12x + 60 \\ \hline & 0 \end{array}$$

$x^2 - x - 12$

$= (x-4)(x+3)$

$\therefore y = (x+5)(x-4)(x+3)$

X int: $-5, 4, -3$



(10) $y = x^3 - 2x^2 - 15x + 36$

$x = 3,\ y = 27 - 18 - 45 + 36 = 0$

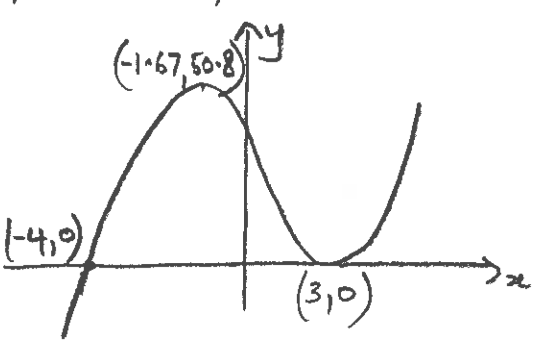
$\therefore\ x - 3$ is a factor

$(x-3)(x^2 + x - 12) = x^3 - 2x^2 - 15x + 36$

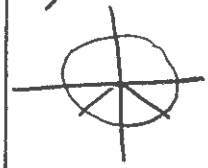
$\therefore y = (x-3)(x+4)(x-3)$

$= (x-3)^2(x+4)$

X int: $x = 3,\ x = -4$



(11) (a) $\sin\theta = -\frac{1}{2}, \quad 0 \leq \theta \leq 2\pi$



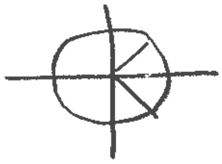
PA $= \frac{\pi}{6}$

$\theta = \pi + \frac{\pi}{6},\ 2\pi - \frac{\pi}{6}$

$= \frac{7\pi}{6},\ \frac{11\pi}{6}$

(b) $\cos 2\theta = \frac{1}{\sqrt{2}} \quad 0 < \theta \leq 2\pi$

$0 \leq 2\theta \leq 4\pi$



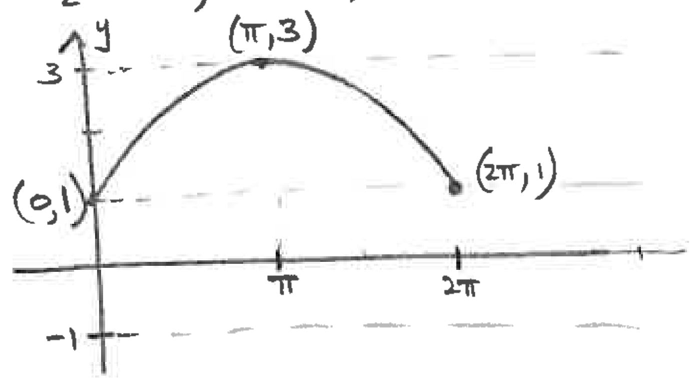
PA $= \frac{\pi}{4}$

$2\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$

$\theta = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

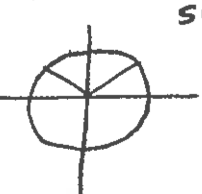
(c) $y = 2\sin\left(\frac{x}{2}\right) + 1 \quad x \in [0, 2\pi]$

$T = \frac{2\pi}{\frac{1}{2}} = 4\pi$, Av $= 1$, Max $= 3$, Min $= -1$



(12) (a) $2\sin(\theta) = 1 \quad \theta \in [0, 2\pi]$

$\sin\theta = \frac{1}{2}$

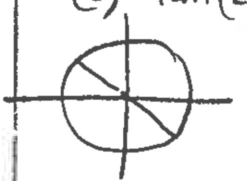


$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

PA $= \frac{\pi}{6}$

(b) $\tan(2\theta) = -\sqrt{3} \quad \theta \in [0, 2\pi]$

$2\theta \in [0, 4\pi]$



PA $= \frac{\pi}{3}$

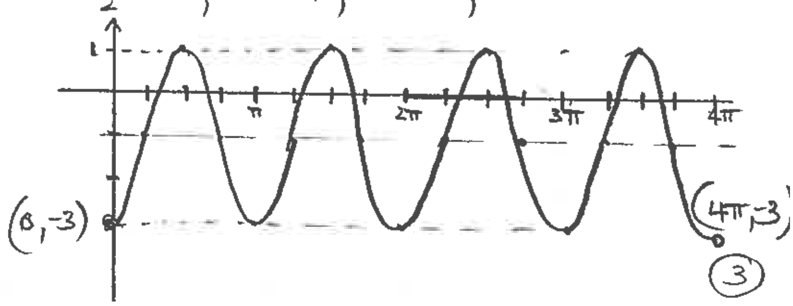
$2\theta = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}, \frac{11\pi}{3}$

$\theta = \frac{2\pi}{6}, \frac{5\pi}{6}, \frac{8\pi}{6}, \frac{11\pi}{6}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$

(c) $y = -2\cos(2x) - 1 \quad 0 \leq x \leq 4\pi$

$T = \frac{2\pi}{2} = \pi$, Av $= -1$, Max $= 1$, Min $= -3$



13) (a) $9^{2x-1} = 3\sqrt{3}$

$(3^2)^{2x-1} = 3 \times 3^{\frac{1}{2}}$

$3^{4x-2} = 3^{3/2}$

$\therefore 4x-2 = \frac{3}{2}$

$8x-4 = 3$

$x = \frac{7}{8}$

(b) $\log_3(x^2-2x-8) = 3$

$x^2-2x-8 = 3^3$

$x^2-2x-8 = 27$

$x^2-2x-35 = 0$

$(x-7)(x+5) = 0$

$x = 7, -5$

(c) $y = 2^{-x} - 8$

Yint: $x=0$, $y = 2^0 - 8 = -7$

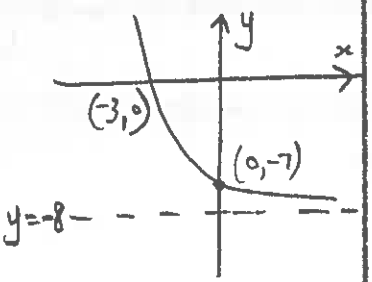
Xint: $2^{-x} - 8 = 0$

$2^{-x} = 8$

$2^{-x} = 2^3$

$-x = 3$

$x = -3$



(d) $y = \log_{10}(4-2x)$

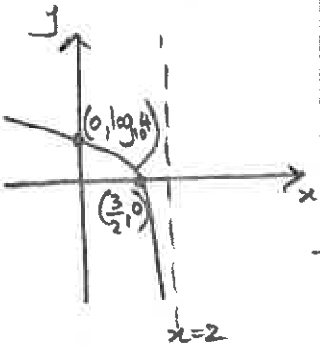
$= \log_{10} -2(x-2)$

Yint: $x=0$, $y = \log_{10}4$

Xint: $\log_{10}(4-2x) = 0$

$4-2x = 1$

$x = \frac{3}{2}$



15) $y = x^2+3x-5$

$\frac{dy}{dx} = 2x+3$

$x=1$, $\frac{dy}{dx} = 2+3 = 5$

$x=1$, $y = 1+3-5 = -1$ $(1,-1)$

$y - y_1 = m(x-x_1)$

$y - -1 = 5(x-1)$

$y+1 = 5x-5$

$y = 5x-6$

(16) $y = 3x^{1/2} - 3x^{-2} + 2$

$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + 6x^{-3}$

$= \frac{3}{2\sqrt{x}} + \frac{6}{x^3}$

$x=1$, $\frac{dy}{dx} = \frac{3}{2} + 6 = \frac{15}{2}$

$x=1$, $y = 3-3+2 = 2$ $(1,2)$

$y - y_1 = m(x-x_1)$

$y - 2 = \frac{15}{2}(x-1)$

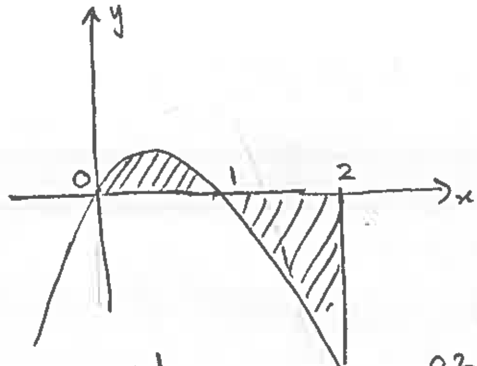
$2y - 4 = 15(x-1)$

$2y-4 = 15x-15$

$15x - 2y = 11$

(17) $y = x(1-x) = x - x^2$

Xint: $x=0$, $x=1$



$\text{Area} = \int_0^1 x - x^2\,dx - \int_1^2 x - x^2\,dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 - \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_1^2$

$= \frac{1}{2} - \frac{1}{3} - \left[\left(\frac{4}{2} - \frac{8}{3}\right) - \left(\frac{1}{2} - \frac{1}{3}\right)\right]$

$= \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} - 2 + \frac{8}{3} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$

$= -1 - \frac{2}{3} + \frac{8}{3}$

$= 1$

(18) $y = x^2 - 1 = (x-1)(x+1)$

$A = -\int_0^1 x^2 - 1\,dx + \int_1^2 x^2 - 1\,dx$

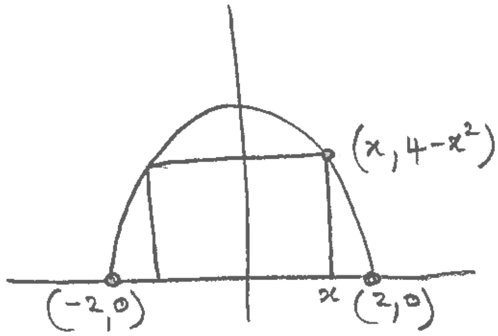
$= -\left[\frac{x^3}{3} - x\right]_0^1 + \left[\frac{x^3}{3} - x\right]_1^2$

$= -\left[\frac{1}{3} - 1\right] + \left[\frac{8}{3} - 2 - \left(\frac{1}{3} - 1\right)\right]$

$= \frac{2}{3} + \frac{4}{3}$

$= 2$

(19) $f(x) = 4 - x^2 = (2-x)(2+x)$



(a) $A = L \times W$

$= 2x \times (4 - x^2)$

$= 2x(4-x^2) \quad , x > 0$

Let $x = \frac{3}{2}$ (Base width $= 3$)

$A = 2 \times \frac{3}{2}\left(4 - \left(\frac{3}{2}\right)^2\right)$

$= 3\left(4 - \frac{9}{4}\right)$

$= 3\left(\frac{16-9}{4}\right)$

$= \frac{21}{4}$

(b) $A = 8x - 2x^3$

$\frac{dA}{dx} = 8 - 6x^2$

Let $\frac{dA}{dx} = 0$

$\therefore 8 - 6x^2 = 0$

$x^2 = \frac{4}{3}$

$x = \frac{2}{\sqrt{3}} \quad , x > 0$

Consider gradient chart

$x = 0, \frac{dA}{dx} = 8 > 0$

$x = 2, \frac{dA}{dx} = 8 - 24 < 0$

$\therefore A$ is max when $x = \frac{2}{\sqrt{3}}$

$x = \frac{2}{\sqrt{3}}, A = 8 \times \frac{2}{\sqrt{3}} - 2 \times \left(\frac{2}{\sqrt{3}}\right)^3$

$= \frac{16}{\sqrt{3}} - \frac{16}{3\sqrt{3}}$

$= \frac{32}{3\sqrt{3}}$

Max Area $= \frac{32}{3\sqrt{3}}$

(c) $A = 2\int_0^2 4 - x^2 \, dx$

$= 2\left[4x - \frac{x^3}{3}\right]_0^2$

$= 2\left[8 - \frac{8}{3}\right]$

$= \frac{32}{3}$

$\therefore$ Area(Rectangle) $= \frac{1}{3} \times \frac{32}{3} = \frac{32}{9}$

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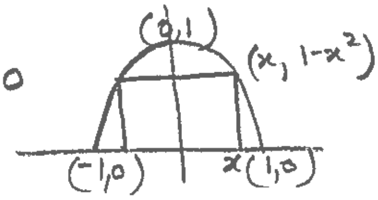
(20) $f(x) = 1 - x^2$



(a) $A = 2x(1-x^2) \quad , x > 0$

$= 2x - 2x^3$

Base width $= 1$

$\Rightarrow x = \frac{1}{2}$

$\therefore A = 2\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^3$

$= 1 - \frac{1}{4}$

$= \frac{3}{4}$

(b) $\frac{dA}{dx} = 2 - 6x^2$

Let $2 - 6x^2 = 0$

$x = \frac{1}{\sqrt{3}} \quad , x > 0$

$x = 0, \frac{dA}{dx} = 2 > 0$

$x = 1 \quad \frac{dA}{dx} = 2 - 6 < 0$

$\therefore A$ is max when $x = \frac{1}{\sqrt{3}}$

$A_{max} = 2\left(\frac{1}{\sqrt{3}}\right) - 2\left(\frac{1}{\sqrt{3}}\right)^3$

$= \frac{2}{\sqrt{3}} - \frac{2}{3\sqrt{3}}$

$= \frac{4}{3\sqrt{3}}$

(c) $A = 2\int_0^1 1 - x^2 \, dx$

$= 2\left[x - \frac{x^3}{3}\right]_0^1$

$= 2\left[1 - \frac{1}{3}\right]$

$= \frac{4}{3}$

A(Rectangle) $= \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$