Exponential & Logs

(1) (a) $4^{1/2} = \sqrt{4} = 2$

(b) $2^{-4} = \frac{1}{2^4} = \frac{1}{16}$

(c) $27^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$

(d) $25^{-3/2} = \frac{1}{25^{3/2}} = \frac{1}{(\sqrt{25})^3} = \frac{1}{5^3}$

$= \frac{1}{125}$

(2) (a) $\frac{(3xy^3)^2}{6xy} \div \frac{2xy^3}{3xy^{-3}}$

$= \frac{9x^2y^6}{6xy} \times \frac{3xy^{-3}}{2xy^3}$

$= \frac{\cancel{27}^{9} x^3 y^{6-3}}{4\,\cancel{12} x^2 y^4}$

$= \frac{9x^3y^3}{4x^2y^4}$

$= \frac{9x}{4y}$

(b) $\frac{xy\sqrt{y}}{2y^2} \times \frac{4x\sqrt[3]{xy}}{x^2y}$

$= \frac{xyy^{\frac{1}{2}}}{2y^2} \times \frac{4x^3x^{\frac{1}{2}}y^{\frac{1}{2}}}{x^2y}$

$= \frac{4x^{1+3+\frac{1}{2}}y^{1+\frac{1}{2}+\frac{1}{2}}}{2y^3x^2}$

$= \frac{2x^{9/2}y^2}{y^3x^2}$

$= \frac{2x^{5/2}}{y}$

$= \frac{2\sqrt{x^5}}{y}$

(c) $\frac{2\sqrt{10}}{\sqrt{5}\sqrt[3]{2}}$

$= \frac{2\sqrt{5}\sqrt{2}}{5^{1/2} \cdot 2^{1/3}}$

$= \frac{2 \times 5^{\frac{1}{2}} \times 2^{\frac{1}{2}}}{5^{1/2} \times 2^{1/3}}$

$= 2^{1+\frac{1}{2}-\frac{1}{3}}$

$= 2^{\frac{6+3-2}{6}}$

$= 2^{7/6}$

3 (a) $2 \times 3^x = 18$

$3^x = 9$

$3^x = 3^2$

$x = 2$

(b) $4 \times \left(\frac{1}{3}\right)^x = 36$

$\left(\frac{1}{3}\right)^x = 9$

$(3^{-1})^x = 3^2$

$3^{-x} = 3^2$

$-x = 2$

$x = -2$

(c) $3^{x-2} = 9^{x-3}$

$3^{x-2} = (3^2)^{x-3}$

$3^{x-2} = 3^{2x-6}$

$x - 2 = 2x - 6$

$x = 4$

(d) $25^{x-2} = \sqrt{5}$

$(5^2)^{x-2} = 5^{\frac{1}{2}}$

$5^{2x-4} = 5^{\frac{1}{2}}$

$2x - 4 = \frac{1}{2}$

$2x = \frac{9}{2}$

$x = \frac{9}{4}$

3(e) $8^{1-2x} = 4^{x-3}\sqrt{8}$

$$(2^3)^{1-2x} = (2^2)^{x-3}\sqrt{2^3}$$

$$2^{3-6x} = 2^{2x-6} \times 2^{\frac{3}{2}}$$

$$2^{3-6x} = 2^{2x-6+\frac{3}{2}}$$

$$3-6x = 2x - \frac{9}{2}$$

$$3 + \frac{9}{2} = 8x$$

$$\frac{15}{2} = 8x$$

$$x = \frac{15}{16}$$

(f) $16^x - 4 \times 8^x = 0$

$$2^{4x} - 2^2 \times 2^{3x} = 0$$

$$2^{3x}(2^x - 2^2) = 0$$

$$2^{3x} = 0 \text{ or } 2^x = 2^2$$

But $2^{3x} > 0$

$$\therefore 2^x = 2^2$$

$$x = 2$$

(g) $10^{2x} - 11 \times 10^x + 10 = 0$

Let $m = 10^x$

$$\therefore m^2 - 11m + 10 = 0$$

$$(m-10)(m-1) = 0$$

$$m = 10 \text{ or } m = 1$$

$$10^x = 10 \text{ or } 10^x = 1$$

$$x = 1 \text{ or } 0$$

(h) $4 \times 2^{2x} + 8 = 33 \times 2^x$

$$4 \times 2^{2x} - 33 \times 2^x + 8 = 0$$

Let $m = 2^x$

$$\therefore 4m^2 - 33m + 8 = 0$$

$$(4m-1)(m-8) = 0$$

$$m = \frac{1}{4} \text{ or } m = 8$$

$$2^x = 2^{-2} \text{ or } 2^x = 2^3$$

$$x = -2 \text{ or } x = 3$$

(i) $3^{2x} - 6 \times 3^x = 27$

Let $m = 3^x$

$$\therefore m^2 - 6m - 27 = 0$$

$$(m-9)(m+3) = 0$$

$$m = 9 \text{ or } m = -3$$

$$3^x = 3^2 \text{ or } 3^x = -3$$

$$\therefore \boxed{x = 2}$$

Not possible $3^x > 0$

(4) $3^n + 3^n + 3^n = 9^n$

$$3 \times 3^n = 9^n$$

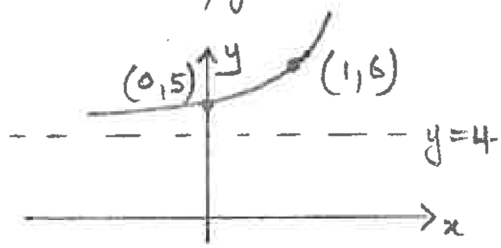
$$3^{n+1} = 3^{2n}$$

$$n + 1 = 2n$$

$$n = 1$$

(5) (a) $y = 2^x + 4$

Yint: $x = 0,\ y = 2^0 + 4 = 5$



dom = $\mathbb{R}$

ran = $(4, \infty)$

$x = 1,\ y = 2^1 + 4 = 6$

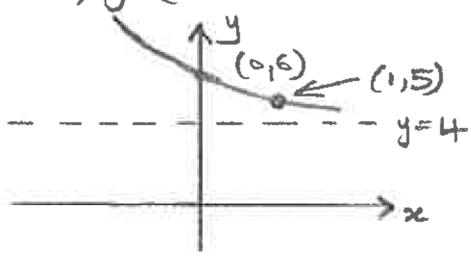
(b) $y = \left(\frac{1}{2}\right)^{x-1} + 4$

Yint: $x = 0,\ y = \left(\frac{1}{2}\right)^{-1} + 4 = 2 + 4 = 6$

Let $x = 1,\ y = \left(\frac{1}{2}\right)^0 + 4 = 5$

dom = $\mathbb{R}$

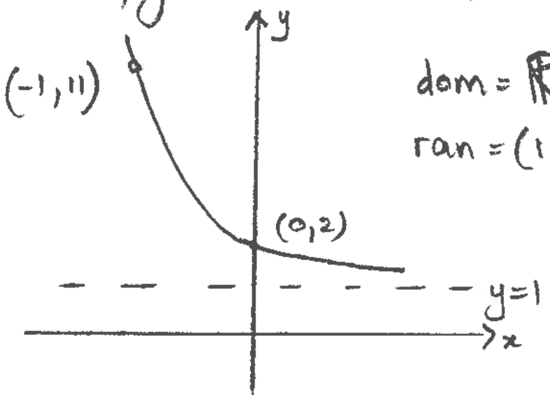
ran = $(4, \infty)$



(c) $y = 10^{-x} + 1$

Y int : $x = 0$, $y = 10^0 + 1 = 2$

$x = -1$, $y = 10^1 + 1 = 11$ $(-1, 11)$



dom = $\mathbb{R}$

ran = $(1, \infty)$

(d) $y = -3^{-x} + 9$

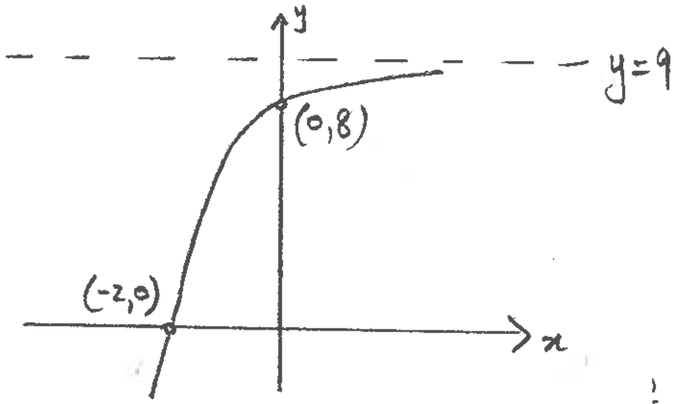
Y int : $x = 0$, $y = -3^0 + 9 = 8$

X int : $-3^{-x} + 9 = 0$

$3^{-x} = 9$

$\therefore 3^{-x} = 3^2$

$x = -2$



dom = $\mathbb{R}$, ran = $(-\infty, 9)$

(e) $f : [-2, \infty) \to \mathbb{R}$, $f(x) = \left(\frac{1}{3}\right)^{x+1} - 3$

Y int : $x = 0$, $y = \left(\frac{1}{3}\right)^1 - 3 = -\frac{8}{3}$

X int : $y = 0$, $\left(\frac{1}{3}\right)^{x+1} - 3 = 0$

$(3^{-1})^{x+1} = 3$

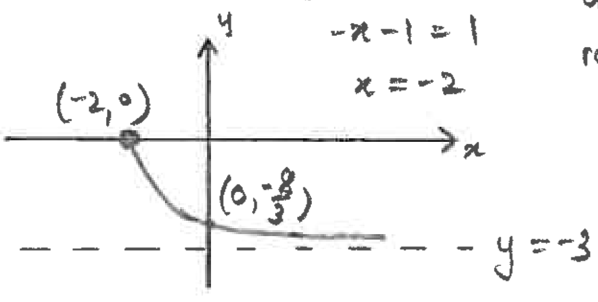
$3^{-x-1} = 3$

$-x - 1 = 1$

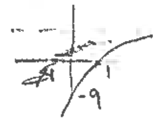
$x = -2$

dom = $[-2, \infty)$

ran = $(-3, 0]$



(f) $y = -10^{-x+1} + 1$



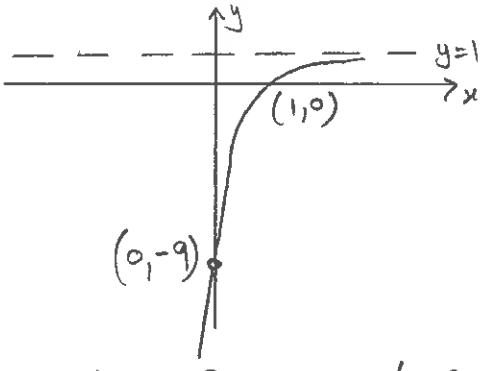
Y int : $x = 0$, $y = -10 + 1 = -9$

X int : $y = 0$, $-10^{-x+1} + 1 = 0$

$10^{-x+1} = 1$

$-x + 1 = 0$

$x = 1$



dom = $\mathbb{R}$, ran = $(-\infty, 1)$

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6 (a) $\log_2 4 = 2$ | $= x$, $2^x = 4$, $x = 2$

(b) $\log_3 9 = 2$ | $= x$, $3^x = 9$, $x = 2$

(c) $\log_3 27 = 3$ | $= x$, $3^x = 27$, $x = 3$

(d) $\log_4 1 = 0$ | $= x$, $4^x = 1$, $x = 0$

(e) $\log_5 5 = 1$ | $= x$, $5^x = 5$, $x = 1$

(f) $\log_{10} 10000 = 4$ | $= x$, $10^x = 10000$, $x = 4$

(g) $\log_3 3^6 = 6$

$= x$

$3^x = 3^6$

$x = 6$

(h) $\log_2 \left(\frac{1}{16}\right) = -4$

$= x$

$2^x = \frac{1}{16}$

$2^x = 2^{-4}$

$x = -4$

(i) $\log_9 \sqrt{3} = \frac{1}{4}$

$= x$

$9^x = \sqrt{3}$

$3^{2x} = 3^{\frac{1}{2}}$

$2x = \frac{1}{2}$

$x = \frac{1}{4}$

(j) $\log_{25} 5 = \frac{1}{2}$

$= x$

$25^x = 5$

$5^{2x} = 5^1$

$2x = 1$

$x = \frac{1}{2}$

(k) $\log_4 \frac{1}{4^4} = -4$

$= x$

$4^x = \frac{1}{4^4}$

$4^x = 4^{-4}$

$x = -4$

(l) $\log_4 \frac{1}{\sqrt{2}} = -\frac{1}{4}$

$= x$

$4^x = \frac{1}{\sqrt{2}}$

$2^{2x} = 2^{-\frac{1}{2}}$

$2x = -\frac{1}{2}$

$x = -\frac{1}{4}$

(7) (a) $3^x = 5$

$x = \log_3 5$

(b) $5^{x+2} = 10$

$x + 2 = \log_5 10$

$x = \log_5 10 - 2$

(c) $3^{2x} - 11 \times 3^x + 18 = 0$

Let $m = 3^x$

$m^2 - 11m + 18 = 0$

$(m-9)(m-2) = 0$

$m = 9, 2$

$\therefore 3^x = 9$ or $3^x = 2$

$x = 2$ or $x = \log_3 2$

(8) (a) $\log_{15} 3 + \log_{15} 5$

$= \log_{15} (3 \times 5)$

$= \log_{15} 15$

$= 1$

(b) $\log_6 3 + \log_6 12$

$= \log_6 36$

$= \log_6 6^2$

$= 2\log_6 6$

$= 2$

(c) $\log_5 10 - \log_5 2$

$= \log_5 \left(\frac{10}{2}\right)$

$= \log_5 5$

$= 1$

(d) $\log_4 48 - \log_4 3$

$= \log_4 16$

$= \log_4 4^2$

$= 2\log_4 4$

$= 2$

(e) $\log_2 54 - 3\log_2 3$

$= \log_2 54 - \log_2 3^3$

$= \log_2 \left(\frac{54}{27}\right)$

$= \log_2 2 = 1$

8(f) $\log_3 225 - 2\log_3 5$

$= \log_3 225 - \log_3 5^2$

$= \log_3 225 - \log_3 25$

$= \log_3 \left(\frac{225}{25}\right)$

$= \log_3 9$

$= \log_3 3^2$

$= 2$

(g) $\frac{\log_3 8}{\log_3 2}$

$= \frac{\log_3 2^3}{\log_3 2}$

$= \frac{3\log_3 2}{\log_3 2}$

$= 3$

(h) $\frac{\log_2 49}{\log_2 7}$

$= \frac{\log_2 7^2}{\log_2 7}$

$= \frac{2\log_2 7}{\log_2 7}$

$= 2$

(i) $\log_5 100 - \frac{1}{2}\log_5 16$

$= \log_5 100 - \log_5 16^{\frac{1}{2}}$

$= \log_5 100 - \log_5 4$

$= \log_5 \left(\frac{100}{4}\right)$

$= \log_5 25$

$= 2$

(j) $\log_3 18 - \frac{1}{3}\log_3 8$

$= \log_3 18 - \log_3 8^{\frac{1}{3}}$

$= \log_3 18 - \log_3 2$

$= \log_3 \left(\frac{18}{2}\right)$

$= \log_3 9$

$= 2$

(k) $\log_2 25 - \log_2 5$

$= \log_2 5$

(l) $\log_6 8 - \log_6 2$

$= \log_6 \frac{8}{2}$

$= \log_6 4$

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(9) Let $\log_b x = m$

$\therefore b^m = x$

Let $\log_b y = n$

$\therefore b^n = y$

$\Rightarrow \frac{x}{y} = \frac{b^m}{b^n}$

$\frac{x}{y} = b^{m-n}$

$\therefore \log_b \left(\frac{x}{y}\right) = m - n$

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(10)

(a) $\log_4 100 - \log_4 2 - \frac{1}{2}\log_4 25$

$= \log_4 100 - \log_4 2 - \log_4 25^{\frac{1}{2}}$

$= \log_4 100 - \log_4 2 - \log_4 5$

$= \log_4 \frac{100}{2 \times 5}$

$= \log_4 10$

(b) $2\log_3 x - 3\log_3 2x + \log_3 9x$

$= \log_3 x^2 - \log_3 (2x)^3 + \log_3 9x$

$= \log_3 x^2 - \log_3 8x^3 + \log_3 9x$

$= \log_3 \frac{x^2 \times 9x}{8x^3}$

$= \log_3 \left(\frac{9}{8}\right)$

(c) $3\log_5 y - \frac{1}{2}\log_5 9x^6 - \log_5 5x$

$= \log_5 y^3 - \log_5 (9x^6)^{\frac{1}{2}} - \log_5 5x^3$

$= \log_5 y^3 - \log_5 3x^3 - \log_5 5x^3$

$= \log_5 \frac{y^3}{3x^3 \times 5x^3}$

$= \log_5 \left(\frac{y^3}{15x^6}\right)$

(d)

$3\log_2 y + \frac{1}{2}\log_2 x^4 - 2\log_2 2xy^2 + 3$

$= \log_2 y^3 + \log_2 x^2 - \log_2 (2xy^2)^2 + 3\log_2$

$= \log_2 y^3 + \log_2 x^2 - \log_2 4x^2y^4 + \log_2 8$

$= \log_2 \frac{y^3 \times x^2 \times 8}{4x^2y^4}$

$= \log_2 \left(\frac{2}{y}\right)$

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(11) $\log_2 a = x$, $\log_2 b = y$

(a) $\log_2 ab$

$= \log_2 a + \log_2 b$

$= x + y$

(b) $\log_2 \left(\frac{a}{b}\right)$

$= \log_2 a - \log_2 b$

$= x - y$

(c) $\log_2 \left(\frac{4a^2}{b}\right)$

$= \log_2 4 + \log_2 a^2 - \log_2 b$

$= 2 + 2\log_2 a - \log_2 b$

$= 2 + 2x - y$

d) $\log_2\left(\frac{8b}{\sqrt{a}}\right)$

$= \log_2 8 + \log_2 b - \log_2 a^{1/2}$

$= 3 + y - \frac{1}{2}\log_2 a$

$= 3 + y - \frac{1}{2}x$

(e) $\log_2\left(\sqrt{\frac{2b^3}{a^2}}\right)$

$= \log_2\left(\frac{2b^3}{a^2}\right)^{1/2}$

$= \frac{1}{2}\log_2\left(\frac{2b^3}{a^2}\right)$

$= \frac{1}{2}\left[\log_2 2 + \log_2 b^3 - \log_2 a^2\right]$

$= \frac{1}{2}\left[1 + 3\log_2 b - 2\log_2 a\right]$

$= \frac{1}{2}\left[1 + 3y - 2x\right]$

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12 (a) $\cancel{d}\ a = b\log_2 N$

$\frac{a}{b} = \log_2 N$

$2^{a/b} = N$

(b) $d = 2 - \log_3 M$

$\log_3 M = 2 - d$

$M = 3^{2-d}$

(c) $\log_2 N = 3\log_2 d - 3$

$3 = \log_3 d^3 - \log_2 N$

$3 = \log_3 \frac{d^3}{N}$

$3^3 = \frac{d^3}{N}$

$27N = d^3$

(d) $\log_{10} P = \frac{1}{2}\log_{10} N$

$\log_{10} P = \log_{10} N^{1/2}$

$P = N^{1/2}$

$P = \sqrt{N}$

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(13)

(a) $\log_{10} x + \log_{10}(x+1) = \log_{10} 30$

$\log_{10} x(x+1) = \log_{10} 30$

$x(x+1) = 30$

$x^2 + x - 30 = 0$

$(x+6)(x-5) = 0$

$x = -6$ or $x = 5$

But $x > 0$

$\therefore x = 5$

(b) $\log_5 x = \log_5 8 - \log_5(6-x)$

$\log_5 x + \log_5(6-x) = \log_5 8$

$\log_5 x(6-x) = \log_5 8$

$x(6-x) = 8$

$6x - x^2 = 8$

$x^2 - 6x + 8 = 0$

$(x-4)(x-2) = 0$

$x = 4, 2$

(c) $\log_2(x+1) - \log_2(x-1) = \frac{3}{4}$

$\log_2 \frac{x+1}{x-1} = \frac{3}{4}$

$\frac{x+1}{x-1} = 2^{3/4}$

$x + 1 = 2^{3/4}(x-1)$

$x + 1 = 2^{3/4}x - 2^{3/4}$

$2^{3/4} + 1 = 2^{3/4}x - x$

$2^{3/4} + 1 = x\left(2^{3/4} - 1\right)$

$x = \frac{2^{3/4} + 1}{2^{3/4} - 1}$

13 (d) $\log_8 \sqrt[4]{x^2+7} = \frac{1}{3}$

$$\log_8 (x^2+7)^{1/4} = \frac{1}{3}$$

$$(x^2+7)^{1/4} = 8^{1/3}$$

$$(x^2+7)^{1/4} = 2$$

$$x^2+7 = 2^4$$

$$x^2+7 = 16$$

$$x^2 = 9$$

$$x = \pm 3$$

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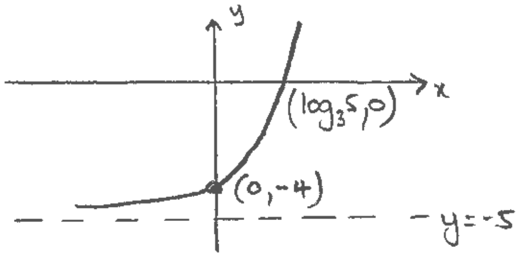
14 (a) $y = 3^x - 5$

Y int: $x=0$, $y = 3^0 - 5 = -4$

X int: $3^x - 5 = 0$

$$3^x = 5$$

$$x = \log_3 5$$



dom $= \mathbb{R}$, ran $= (-5, \infty)$

(b) $f: [-3,\infty) \to \mathbb{R}$, $f(x) = -2^{-x} + 6$

Y int: $x=0$, $f(0) = -2^0 + 6 = 5$

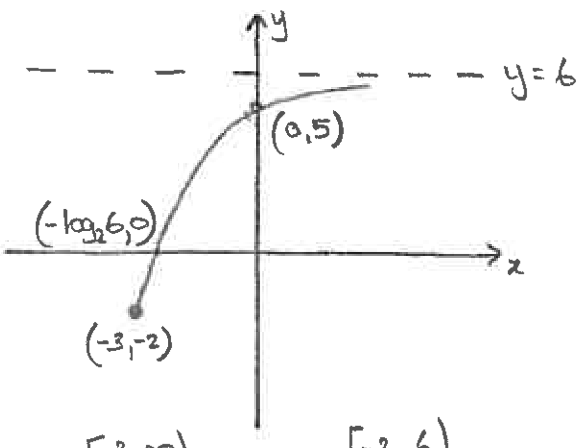
X int: $-2^{-x} + 6 = 0$

$$2^{-x} = 6$$

$$-x = \log_2 6$$

$$x = -\log_2 6$$

$x=-3$, $f(-3) = -2^{--3} + 6 = -8+6 = -2$



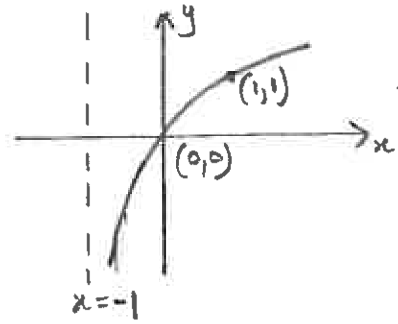
dom $= [-3, \infty)$, ran $= [-2, 6)$

15

(a) $y = \log_2(x+1)$

Y int: $x=0$, $y = \log_2 1 = 0$

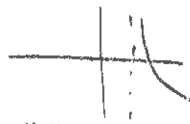
Let $x=1$, $y = \log_2 2 = 1$ (1,1)



dom $= (-1, \infty)$, ran $= \mathbb{R}$

(b) $y = -\log_{10}(2x-2)$

$$y = -\log_{10}(2(x-1))$$



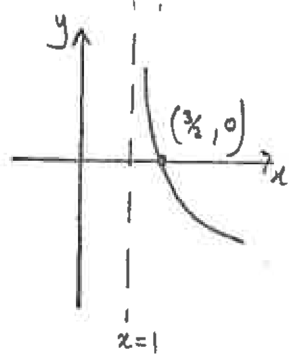
X int: $-\log_{10}(2x-2) = 0$

$$\log_{10}(2x-2) = 0$$

$$2x - 2 = 10^0$$

$$2x = 3$$

$$x = 3/2$$



dom $= (1, \infty)$, ran $= \mathbb{R}$

(c) $y = -2\log_2(x+2)$

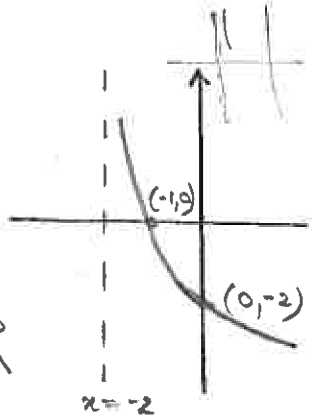
Y int: $y = -2\log_2 2 = -2$

X int: $-2\log_2(x+2) = 0$

$$\log_2(x+2) = 0$$

$$x+2 = 1$$

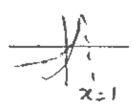
$$x = -1$$



dom $= (-2, \infty)$, ran $= \mathbb{R}$

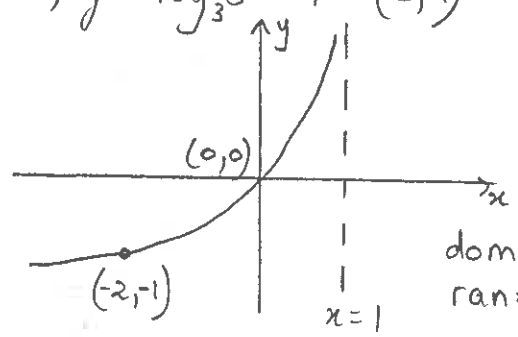
(d) $y = -\log_3(1-x)$

$$\therefore y = -\log_3(-(x-1))$$



$x=0$, $y = -\log_3 1 = 0$ (0,0)

$x=-2$, $y = -\log_3 3 = -1$ (-2,-1)



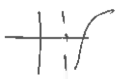
dom $= (-\infty, 1)$

ran $= \mathbb{R}$

(e) $y = \log_{10}(3x-6) + 2$

$y = \log_{10}(3(x-2)) + 2$

$3x - 6 = 10$
$x = \frac{16}{3}$



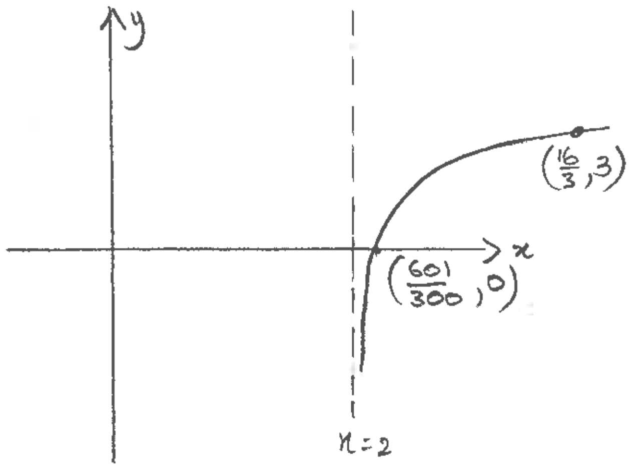
X int: $\log_{10}(3x-6) + 2 = 0$

$\log_{10}(3x-6) = -2$

$3x - 6 = 10^{-2}$ $\quad 6\frac{1}{100}$

$3x = \frac{601}{100}$

$x = \frac{601}{300}$



$x = \frac{16}{3}$, $y = \log_{10} 10 + 2 = 3$

dom $= (2, \infty)$, ran $= \mathbb{R}$

(f) $f : (-5, 0) \to \mathbb{R}$, $f(x) = -2\log_5(-x) - 1$

X int: $-2\log_5(-x) - 1 = 0$

$-2\log_5(-x) = 1$

$\log_5(-x) = -\frac{1}{2}$

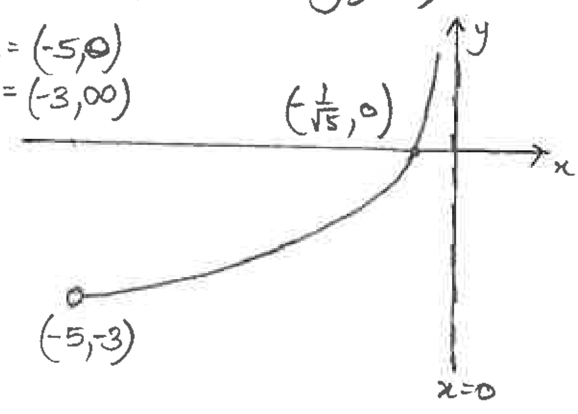
$-x = 5^{-1/2}$

$-x = \frac{1}{\sqrt{5}}$

$x = -\frac{1}{\sqrt{5}}$

Let $x = -5$, $f(-5) = -2\log_5(-(-5)) - 1 = -2 - 1 = -3$

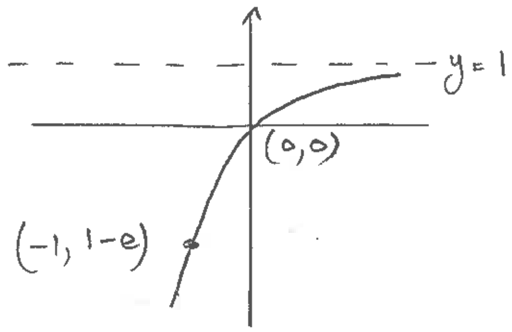
dom $= (-5, 0)$
ran $= (-3, \infty)$



(16) (a) $y = -e^{-x} + 1$

Y int: $x = 0$, $y = -e^0 + 1 = 0$

$x = -1$, $y = -e^{-(-1)} + 1 = 1 - e$

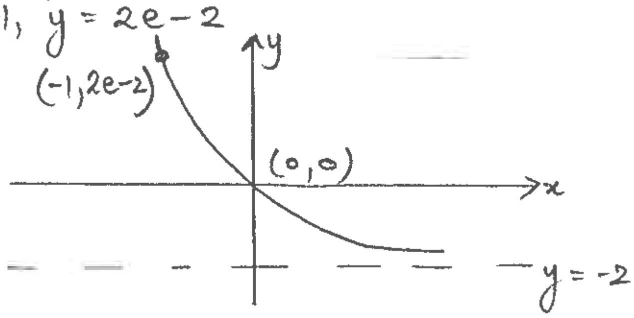


dom $= \mathbb{R}$, ran $= (-\infty, 1)$

(b) $y = 2e^{-x} - 2$

$x = 0$, $y = 2e^0 - 2 = 0$

$x = -1$, $y = 2e - 2$



dom $= \mathbb{R}$, ran $= (-2, \infty)$

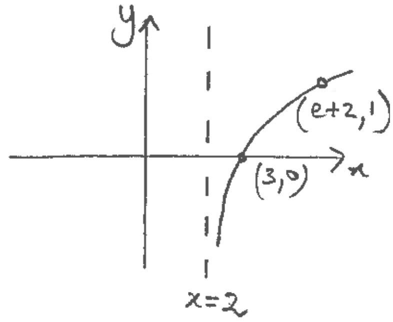
(c) $y = \log_e(x-2)$

X int: $\log_e(x-2) = 0$

$x - 2 = 1$

$x = 3$

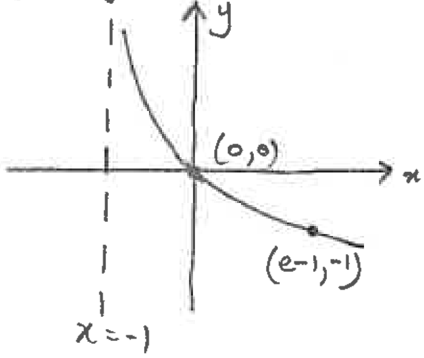
dom $= (2, \infty)$
ran $= \mathbb{R}$



(d) $y = -\log_e(x+1)$

$x = 0$, $y = -\log_e 1 = 0$

$x = e - 1$, $y = -\log_e e = -1$



(17) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 10^{x+1} + 1$

$x = 0$, $f(0) = 10^1 + 1 = 11$

$x = 1$, $f(1) = 10^2 + 1 = 101$

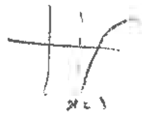
To find inverse switch x and y

$$x = 10^{y+1} + 1$$

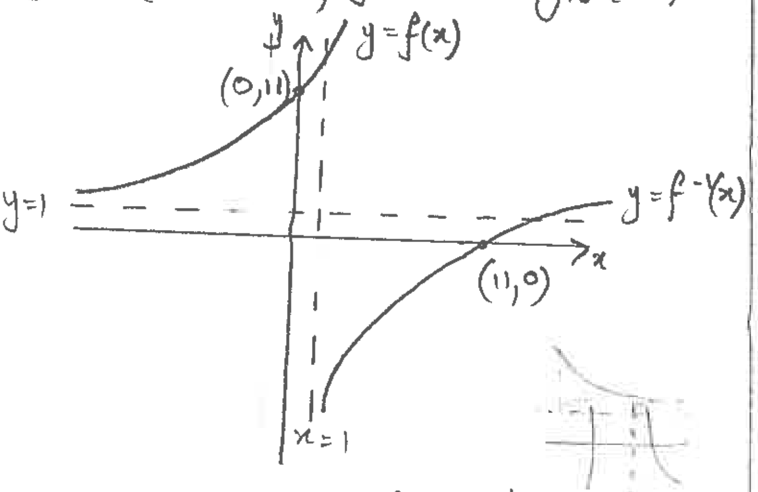
$$x - 1 = 10^{y+1}$$

$$y + 1 = \log_{10}(x-1)$$

$$y = \log_{10}(x-1) - 1$$



$f^{-1}: (1, \infty) \to R$, $f^{-1}(x) = \log_{10}(x-1) - 1$



(b) $f: (2, \infty) \to \mathbb{R}$, $f(x) = -\log_e(x-2)$

X int: $-\log_e(x-2) = 0 \Rightarrow x - 2 = 1$, $x = 3$

$x = e+2$, $y = -\log_e e = -1$

To find rule for inverse switch x and y

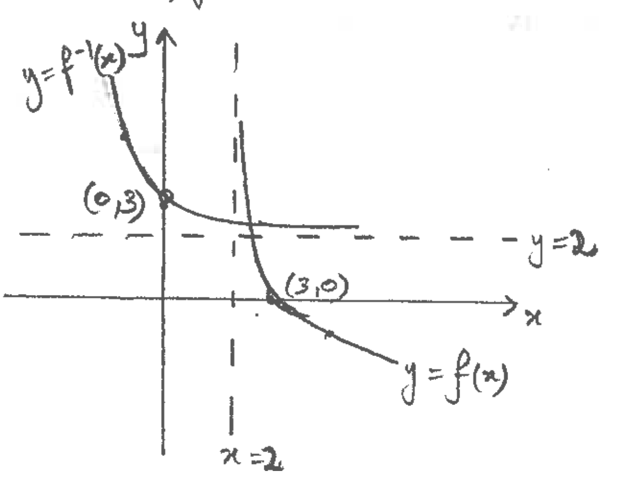
$$x = -\log_e(y-2)$$

$$-x = \log_e(y-2)$$

$$y - 2 = e^{-x}$$

$$y = e^{-x} + 2$$

$f^{-1}: \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = e^{-x} + 2$



(18) (a) $2 \times 3^{2a} + 3 = 5$

$$2 \times 3^{2a} = 2$$

$$3^{2a} = 1$$

$$2a = 0$$

$$a = 0$$

(b) $3 \times \left(\frac{1}{2}\right)^{2b} + 7 = 11$

$$3 \times \left(\frac{1}{2}\right)^{2b} = 4$$

$$\left(\frac{1}{2}\right)^{2b} = \frac{4}{3}$$

$$2b = \log_{\frac{1}{2}}\left(\frac{4}{3}\right)$$

$$b = \frac{1}{2}\log_{\frac{1}{2}}\left(\frac{4}{3}\right)$$

$$b = \log_{\frac{1}{2}}\left(\frac{4}{3}\right)^{\frac{1}{2}}$$

$$b = \log_{\frac{1}{2}}\sqrt{\frac{4}{3}}$$

(c) $b \times 10^{4a} = 6$ ①

$b \times 10^{2a} = 3$ ②

① ÷ ② $\dfrac{b \times 10^{4a}}{b \times 10^{2a}} = \dfrac{6}{3}$

$$10^{2a} = 2$$

$$2a = \log_{10} 2$$

$$a = \frac{1}{2}\log_{10} 2$$

Sub into ①

$$b \times 10^{4 \times \frac{1}{2}\log_{10} 2} = 6$$

$$b \times 10^{2\log_{10} 2} = 6$$

$$b \times 10^{\log_{10} 4} = 6$$

$$4b = 6$$

$$b = \frac{6}{4} = \frac{3}{2}$$

$a = \frac{1}{2}\log_{10} 2$, $b = \frac{3}{2}$

18(d) $b \times 2^{2a} = 4$ ①

$b \times 2^{-2a} = 2$ ②

① ÷ ② $\frac{b \times 2^{2a}}{b \times 2^{-2a}} = \frac{4}{2}$

$2^{4a} = 2$

$4a = 1$

$a = \frac{1}{4}$

① $b \times 2^{2 \times \frac{1}{4}} = 4$

$b \times 2^{1/2} = 4$

$b = \frac{4}{2^{1/2}} = \frac{2^2}{2^{1/2}} = 2^{3/2} = \sqrt{2^3}$

$= \sqrt{8} = 2\sqrt{2}$

$a = \frac{1}{4}, \; b = 2\sqrt{2}$

---

(19) $W = 5 \times e^{0.08t}$

(a) $t=0, \; W = 5 \times e^0 = 5$

Let $W = 10$

$5 \times e^{0.08t} = 10$

$e^{0.08t} = 2$

$0.08t = \log_e 2$

$\frac{2t}{25} = \log_e 2$

$t = \frac{25}{2} \log_e 2$

$0.08 = \frac{8}{100} = \frac{2}{25}$

(b) $t=10, \; W = 5 \times e^{0.08 \times 10}$

$= 5e^{0.8}$

$\simeq 11.1$ grams

(c) $5e^{0.08t} = 6$

$e^{0.08t} = \frac{6}{5}$

$0.08t = \log_e \left(\frac{6}{5}\right)$

$t = \frac{25}{2} \log_e \left(\frac{6}{5}\right)$

$\simeq 2.3$

(20) $W(t) = A \times 2^{-0.001t}$

(a) $t = 0, \; W = A \times 2^0 = A$

(b) Let $W = \frac{A}{2}$

$A \times 2^{-0.001t} = \frac{A}{2}$

$2^{-0.001t} = \frac{1}{2}$

$2^{-0.001t} = 2^{-1}$

$-0.001t = -1$

$0.001t = 1$

$t = 1000$

1000 years

(c) $t = 200, \; W = A \times 2^{-0.001 \times 200}$

$W = A \times 2^{-0.2}$

$\% \text{ rem} = \frac{A \times 2^{-0.2}}{A} \times 100\%$

$= 2^{-0.2} \times 100\%$

$\simeq 87.1\%$

(d) $\frac{A \times 2^{-0.001t}}{A} \times 100 = P$

$100 \times 2^{-0.001t} = P$

$2^{-0.001t} = \frac{P}{100}$

$-0.001t = \log_2 \left(\frac{P}{100}\right)$

$t = -1000 \log_2 \left(\frac{P}{100}\right)$

$= 1000 \log_2 \left(\frac{100}{P}\right)$

---

(21) (a) $A_0 = 100$

$A_1 = 0.95 \times 100$

$A_2 = 0.95^2 \times 100$

$\therefore A_n = 0.95^n \times 100$

$A_7 = 0.95^7 \times 100$

$\simeq 69.8$

(b) $0.95^n \times 100 = 50$

$0.95^n = \frac{1}{2}$

$n = \log_{0.95} \frac{1}{2}$

$\simeq 13.5$

$13\frac{1}{2}$ years

(22) Let n = number of rolls

$$Pr(\text{At least one } 6) = 0.8$$

$$1 - Pr(\text{No } 6) = 0.8$$

$$Pr(\text{No six}) = 0.2$$

$$\left(\frac{5}{6}\right)^n = 0.2$$

$$n = \log_{5/6} 0.2$$

$$\simeq 8.8$$

At least 9 rolls

---

(23) Let n = number of rolls

$$Pr(\text{At least one six}) = 0.99$$

$$1 - Pr(\text{No six}) = 0.99$$

$$Pr(\text{No six}) = 0.01$$

$$\left(\frac{5}{6}\right)^n = 0.01$$

$$n = \log_{5/6} 0.01$$

$$\simeq 25.3$$

At least 26 rolls

---

(24)(a) $\log_2(\log_2 x) = 3$

$$\log_2 x = 2^3$$

$$\log_2 x = 8$$

$$x = 2^8 = 256$$

(b) $(\log_2 x)^2 = 2\log_2 x$

$$(\log_2 x)^2 - 2\log_2 x = 0$$

$$\log_2 x\,[\log_2 x - 2] = 0$$

$\log_2 x = 0$ or $\log_2 x = 2$

But $\log_2 x \neq 0$

$\therefore \log_2 x = 2$

$$x = 2^2 = 4$$

(25)

$$5^{2a} \times 3^{2a+1} = 10^{2a+1}$$

$$5^{2a} = \frac{10^{2a+1}}{3^{2a+1}}$$

$$5^{2a} = \left(\frac{10}{3}\right)^{2a+1}$$

$$\log_e 5^{2a} = \log_e \left(\frac{10}{3}\right)^{2a+1}$$

$$2a\log_e 5 = (2a+1)\log_e\left(\frac{10}{3}\right)$$

$$2a\log_e 5 = 2a\log_e\left(\frac{10}{3}\right) + \log_e\left(\frac{10}{3}\right)$$

$$2a\log_e 5 - 2a\log_e\left(\frac{10}{3}\right) = \log_e\left(\frac{10}{3}\right)$$

$$2a\left[\log_e 5 - \log_e\left(\frac{10}{3}\right)\right] = \log_e\left(\frac{10}{3}\right)$$

$$2a\log_e\left(5 \div \frac{10}{3}\right) = \log_e\left(\frac{10}{3}\right)$$

$$2a\log_e\left(5 \times \frac{3}{10}\right) = \log_e\left(\frac{10}{3}\right)$$

$$2a\log_e\left(\frac{3}{2}\right) = \log_e\left(\frac{10}{3}\right)$$

$$a = \frac{\log_e\left(\frac{10}{3}\right)}{2\log_e\left(\frac{3}{2}\right)}$$

$$= \frac{\log_e\left(\frac{10}{3}\right)}{\log_e\left(\frac{9}{4}\right)}$$

---

(26) $y = \log_2 x \qquad y = (\log_2 x)^2$

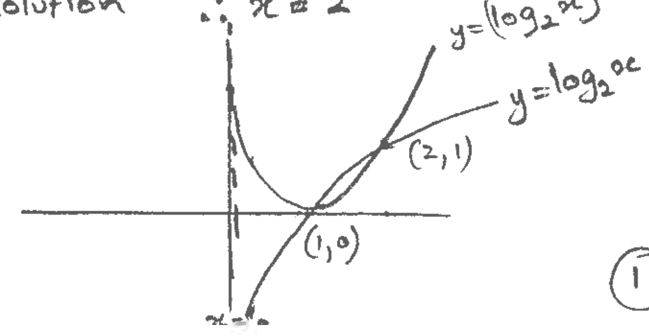
Let $(\log_2 x)^2 = \log_2 x$

$$(\log_2 x)^2 - \log_2 x = 0$$

$$\log_2 x\,[\log_2 x - 1] = 0$$

$\log_2 x = 0$, $\log_2 x = 1$

No solution $\quad \therefore x = 2$



27) $(x^2+3x+3)^{x^2-5x+6}=1$

$\log_{10}(x^2+3x+3)^{x^2-5x+6} = \log_{10} 1$

$(x^2-5x+6)\log_{10}(x^2+3x+3) = 0$

$\therefore x^2-5x+6=0$ or $\log_{10}(x^2+3x+3)=0$

$(x-2)(x-3)=0$ or $x^2+3x+3=1$

$x=2,3$ or $x^2+3x+2=0$

$(x+2)(x+1)=0$

$x=-2,-1$

$x=-2,-1,2,3$

(28) $\log_b x^d = d\log_b x$

Let $\log_b x = m$

$\therefore x = b^m$

$\therefore x^d = (b^m)^d = b^{md}$

$x^d = b^{dm}$

$\log_b x^d = dm$

$\therefore \log_b x^d = d\log_b x$

as required.

(29) (a) Let $x = \log_b a$

$\therefore b^x = a$

$\log_c b^x = \log_c a$

$x\log_c b = \log_c a$

$x = \dfrac{\log_c a}{\log_c b}$

$\therefore \log_b a = \dfrac{\log_c a}{\log_c b}$ as req

(b) $\log_2 5 \log_5 2$

$= \dfrac{\cancel{\log_e 5}}{\log_e 2} \times \dfrac{\log_e 2}{\cancel{\log_e 5}}$

$= 1$

(30) $3^{2x-1} = 4^{3-x}$

$2x-1 = \log_3 4^{3-x}$

$2x-1 = (3-x)\log_3 4$

$2x-1 = 3\log_3 4 - x\log_3 4$

$2x + x\log_3 4 = 3\log_3 4 + 1$

$x(2+\log_3 4) = 3\log_3 4 + 1$

$x = \dfrac{3\log_3 4 + 1}{2+\log_3 4}$

$= \dfrac{\log_3 4^3 + \log_3 3}{2\log_3 3 + \log_3 4}$

$= \dfrac{\log_3 64 + \log_3 3}{\log_3 9 + \log_3 4}$

$= \dfrac{\log_3 192}{\log_3 36}$

$= \log_{36} 192$

(31) $2^{x-1} = 3^{2-x}$

$x-1 = \log_2 3^{2-x}$

$x-1 = (2-x)\log_2 3$

$x-1 = 2\log_2 3 - x\log_2 3$

$x(1+\log_2 3) = 1 + 2\log_2 3$

$x = \dfrac{\log_2 2 + \log_2 9}{\log_2 2 + \log_2 3}$

31 continued)

$$x = \frac{\log_2 18}{\log_2 6} = \log_6 18$$

(32) $x^{\log_b y} = y^{\log_b x}$

Let $x^{\log_b y} = m$

$\therefore \log_b y = \log_x m$

$\therefore y = b^{\log_x m}$

So $y^{\log_b x}$

$= \left(b^{\log_x m}\right)^{\log_b x}$

$= \left(b^{\log_b x}\right)^{\log_x m}$

$= x^{\log_x m}$

$= m$

$= x^{\log_b y}$

as required.

(33) Assume there are two natural numbers m, n such that

$\log_2 3 = \frac{m}{n}$

$\therefore 3 = 2^{m/n}$

$\therefore 3^n = 2^m$

But 3^n is an odd number and 2^m is even

Hence $3^n \neq 2^m$

So there is a contradiction

Assuming $\log_2 3 = \frac{m}{n}$ leads to an impossibility

$\therefore \log_2 3 \neq \frac{m}{n}$

(34) $\frac{1}{2}(x-y)^2 \geq 0$

$\therefore \frac{1}{2}(x^2 - 2xy + y^2) \geq 0$

$\frac{x^2}{2} - xy + \frac{y^2}{2} \geq 0$

$\frac{x^2}{2} + \frac{y^2}{2} \geq xy$ ①

Now $\left(\frac{x+y}{2}\right)^2$

$= \frac{(x+y)^2}{4}$

$= \frac{x^2 + 2xy + y^2}{4}$

$= \frac{x^2}{4} + \frac{xy}{2} + \frac{y^2}{4}$

$= \frac{1}{2}\left(\frac{x^2}{2} + xy + \frac{y^2}{2}\right)$

$\geq \frac{1}{2}(xy + xy)$ using ①

$= xy$

So $xy \leq \left(\frac{x+y}{2}\right)^2$

$\Rightarrow \log_b(xy) \leq \log_b\left(\frac{x+y}{2}\right)^2$

$\Rightarrow \log_b x + \log_b y \leq 2\log_b\left(\frac{x+y}{2}\right)$

$\Rightarrow \frac{\log_b x + \log_b y}{2} \leq \log_b\left(\frac{x+y}{2}\right)$

(35)

$D = 5 + 2 \times \frac{3}{4} \times 5 + 2 \times \left(\frac{3}{4}\right)^2 \times 5 + 2 \times \left(\frac{3}{4}\right)^3 \times 5 + \ldots$ ①

$\frac{3}{4}D = \frac{3}{4} \times 5 + 2 \times \left(\frac{3}{4}\right)^2 \times 5 + 2 \times \left(\frac{3}{4}\right)^3 \times 5 + \ldots$ ②

① − ② : $\frac{1}{4}D = 5 + 2 \times \frac{3}{4} \times 5 - \frac{3}{4} \times 5 + 0 + 0 + \ldots$

$\therefore \frac{1}{4}D = 5 + \frac{3}{4} \times 5$

$\frac{1}{4}D = \frac{20 + 15}{4}$

$D = 35$