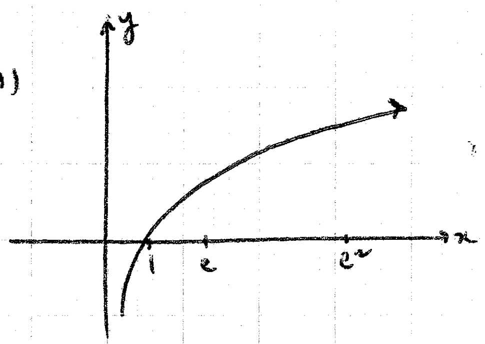
DIFFERENTIATION SOLUTIONS

(Q1) (A)

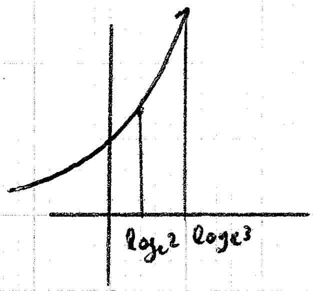


$$\frac{\Delta y}{\Delta x} = \frac{f(e^2) - f(e)}{e^2 - e}$$

$$= \frac{\log_e(e^2) - \log_e(e)}{e^2 - e}$$

$$= \frac{2-1}{e^2-e} = \frac{1}{e^2-e}$$

(B)



$$\frac{\Delta y}{\Delta x} = \frac{f(\log_e 3) - f(\log_e 2)}{\log_e(3) - \log_e 2}$$

$$= \frac{e^{\log_e 3} - e^{\log_e 2}}{\log_e(3/2)}$$

$$= \frac{3-2}{\log_e(3/2)}$$

$$= \frac{1}{\log_e(3/2)}$$

(Q2) (A)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{4(x+h)^2 - 3(x+h) - 1 - (4x^2 - 3x - 1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4x^2} + 8xh + h^2 - \cancel{3x} - 3h \cancel{-1} - \cancel{4x^2} + \cancel{3x} \cancel{+1}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(8x + h - 3)}{\cancel{h}}$$

$$= 8x - 3$$

(B)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)}$$

$$= -\frac{1}{x^2}$$

(Q3) (A) $f(x) = 3x^4 - 2x^2 + 3x + 2$

$$f'(x) = 12x^3 - 4x + 3$$

(B) $f(x) = x^{1/2} + 2x^{1/3} - x^{-1} + x^{-1/2}$

$$f'(x) = \frac{1}{2}x^{-1/2} + \frac{2}{3}x^{-2/3} + x^{-2} - \frac{1}{2}x^{-3}$$

$$= \frac{1}{2\sqrt{x}} + \frac{2}{3x^{2/3}} + \frac{1}{x^2} - \frac{1}{2x^{3/}}$$

#2

(C) $f(x) = 4x^2 + 4x - x^{-1/2}$

$f'(x) = 8x + 4 + \frac{1}{2}x^{-3/2}$

$= 8x + 4 + \frac{1}{2x^{3/2}}$

(D) $f(x) = 4\cos 3x - 3\sin 6x$

$f'(x) = -12\sin 3x - 18\cos 6x$

(E) $y = x^{1/4} - 2e^x$

$\frac{dy}{dx} = \frac{1}{4}x^{-3/4} - 2e^x$

$= \frac{1}{4x^{3/4}} - 2e^x$

(F) $f(x) = 4\log_e x - 4\tan 5x$

$f'(x) = \frac{4}{x} - 20\sec^2(5x)$

(G) $y = \frac{x-3}{\sqrt{x}}$

$= x^{1/2} - 3x^{-1/2}$

$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} + \frac{3}{2}x^{-3/2}$

$= \frac{1}{2\sqrt{x}} + \frac{3}{2x^{3/2}}$

(H) $f(x) = -2\cos(\pi x) + 3\sin(6\pi x)$

$f'(x) = 2\pi\sin(\pi x) + 18\pi\cos(6\pi x)$

(i) $y = \frac{e^{2x} + 2e^x}{e^x}$

$= e^x + 2$

$\frac{dy}{dx} = e^x$

(j) $f(x) = 2\tan x - 3$

$f'(x) = 2\sec^2 x$

(Q4) $f(x) = 3\sin 2x - 3\cos 5x + e^{2x}$

$f'(x) = 6\cos 2x + 15\sin 5x + 2e^{2x}$

$\therefore f'(0) = 6\cos 0 + 15\sin 0 + 2e^0$

$= 6 + 2$

$= 8$

(Q5) $f(x) = 3x^{1/2} - 2x^{-1/3} + \log_e x$

$f'(x) = \frac{3}{2}x^{-1/2} + \frac{2}{3}x^{-4/3} + \frac{1}{x}$

$= \frac{3}{2\sqrt{x}} + \frac{2}{3x^{4/3}} + \frac{1}{x}$

$\therefore f'(1) = \frac{3}{2} + \frac{2}{3} + 1$

$= \frac{9}{6} + \frac{4}{6} + \frac{6}{6}$

$= \frac{19}{6}$

(Q6) $f(x) = ax^3 + x^2 + x$

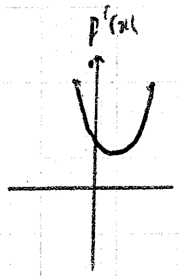
$f'(x) = 3ax^2 + 2x + 1$

$\Delta = 2^2 - 4(3a)(1)$

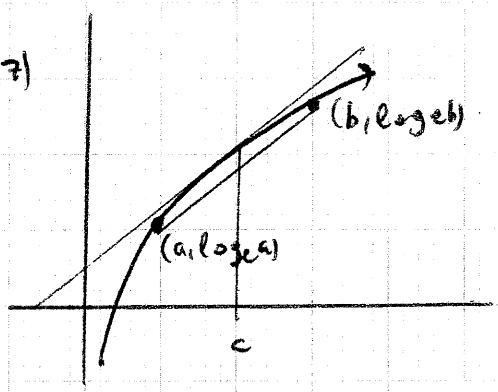
$= 4 - 12a < 0$

$\Leftrightarrow \quad 4 < 12a$

$\Leftrightarrow \quad a > \frac{1}{3}$



(Q7)



$$\frac{f(b) - f(a)}{b - a} = f'(c) = \frac{1}{c}$$

$$\frac{\log_e b - \log_e a}{b - a} = \frac{1}{c}$$

$$\frac{\log_e(b/a)}{b - a} = \frac{1}{c}$$

$$c = \frac{b - a}{\log_e(b/a)}$$

Q8 (A) $y = (4x+3)^4$

$$\frac{dy}{dx} = 4(4x+3)^3 \cdot 4$$

$$= 16(4x+3)^3$$

(B) $f(x) = \cos(3x^2 - 4x)$

$$f'(x) = -\sin(3x^2 - 4x) \times (6x - 4)$$

$$= -(6x-4)\sin(3x^2 - 4x)$$

(C) $y = (e^{2x} - 1)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}}(e^{2x} - 1)^{-1/2} \cdot \cancel{2}e^{2x}$$

$$= \frac{e^{2x}}{\sqrt{e^{2x} - 1}}$$

(D) $f(x) = \tan(\log_e(2x))$

$$f'(x) = \sec^2(\log_e(2x)) \cdot \frac{1}{x}$$

$$= \frac{1}{x}\sec^2(\log_e(2x))$$

(E) $y = (x^2 - x)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 - x)^{-1/2}(2x - 1)$$

$$= \frac{2x - 1}{2\sqrt{x^2 - x}}$$

(F) $f(x) = \sin(\cos x)$

$$f'(x) = \cos(\cos(x))(-\sin x)$$

$$= -\sin x \cos(\cos(x))$$

(G) $y = \log_e(\cos x + \sin x)$

$$\frac{dy}{dx} = \frac{1}{\cos x + \sin x} \cdot (-\sin x + \cos x)$$

$$= \frac{\cos x - \sin x}{\cos x + \sin x}$$

(H) $f(x) = \tan(e^{3x} - x^3)$

$f'(x) = \sec^2(e^{3x} - x^3) \cdot (3e^{3x} - 3x^2)$

$= (3e^{3x} - 3x^2)\sec^2(e^{3x} - x^3)$

(Q9) (A) $y = \sin(\cos(\log_e x))$

$\frac{dy}{dx} = \cos(\cos(\log_e x))$

$\cdot -\sin(\log_e x)$

$\cdot \frac{1}{x}$

$= -\frac{1}{x}\sin(\log_e x)\cos(\cos(\log_e x))$

(B) $f(x) = \cos(\tan e^{x^2})$

$f'(x) = -\sin(\tan e^{x^2})$

$\cdot \sec^2(e^{x^2})$

$\cdot e^{x^2}$

$\cdot 2x$

$= -2xe^{x^2}\sec^2(e^{x^2})\sin(\tan e^{x^2})$

(C) $y = \sin(\cos(\tan(\log_e x)))$

$\frac{dy}{dx} = \cos(\cos(\tan(\log_e x)))$

$\cdot -\sin(\tan(\log_e x))$

$\cdot \sec^2(\log_e x)$

$\cdot \frac{1}{x}$

$= -\frac{1}{x}\sec^2(\log_e x)\sin(\tan(\log_e x))$

$\cdot \cos(\cos(\tan(\log_e x)))$

(D) $f(x) = \sin(\log_e(\cos(2x^3)))$

$f'(x) = \cos(\log_e(\cos(2x^3)))$

$\cdot$

(Q10) $f(2) = e \quad f'(2) = -2$

$g(x) = \log_e f(x)$

$g'(x) = \frac{1}{f(x)} \cdot f'(x)$

$g'(2) = \frac{1}{f(2)} \cdot f'(2)$

$= -\frac{2}{e}$

(Q11) $f(1) = 2 \quad f'(1) = -2$

$g(x) = e^{f(x)}$

$g'(x) = e^{f(x)} \cdot f'(x)$

$g'(1) = e^{f(1)} \cdot f'(1)$

$= e^2 \cdot (-2)$

$= -2e^2.$

(Q12) $f(0) = 3 \quad f'(0) = -4$

$$g(x) = (f(x)+1)^{1/2}$$

$$g'(x) = \frac{1}{2}(f(x)+1)^{-1/2} f'(x)$$

$$g'(0) = \frac{1}{2}(f(0)+1)^{-1/2} f'(0)$$

$$= \frac{1}{2}(3+1)^{-1/2} \cancel{f}'(-4)$$

$$= \frac{1}{2\sqrt{4}} \cdot (-4)$$

$$= \frac{1}{4}(-4)$$

$$= -1$$

(Q13) (A) $y = x \cos x$

$$\frac{dy}{dx} = x \frac{d}{dx}\cos x + \cos x \frac{d}{dx} x$$

$$= -x \sin x + \cos x$$

(B) $f(x) = e^{3x} \sin 2x$

$$f'(x) = e^{3x} \cdot 2\cos 2x + \sin 2x \cdot 3e^{3x}$$

$$= 2e^{3x}\cos 2x + 3e^{3x}\sin 2x$$

(C) $y = \log_e x \cdot (2x-3)^{1/2}$

$$\frac{dy}{dx} = \log_e x \cdot \frac{1}{\cancel{2}}(2x-3)^{-\frac{1}{2}} \cdot \cancel{2} + (2x-3)^{1/2} \cdot \frac{1}{x}$$

$$= \frac{\log_e x}{\sqrt{2x-3}} + \frac{\sqrt{2x-3}}{x}$$

(D) $f(x) = (3x-1)^5 (x+1)^6$

$$f'(x) = (3x-1)^5 \cdot 6(x+1)^5 + (x+1)^6 \cdot 5(3x-1)^4 \cdot 3$$

$$= 6(x+1)^5(3x-1)^5 + 15(3x-1)^4(x+1)^6$$

$$= 3(x+1)^5(3x-1)^4\left(2(3x-1) + 5(x+1)\right)$$

$$= 3(x+1)^5(3x-1)^4(6x - 2 + 5x + 5)$$

$$= 3(x+1)^5(3x-1)^4(11x+3)$$

(E) $y = \sin(\pi x)\cos(2\pi x)$

$$\frac{dy}{dx} = \sin(\pi x) \cdot (-2\pi \sin(2\pi x)) + \cos(2\pi x) + \pi \cos \pi x$$

$$= -2\pi \sin(\pi x)\sin(2\pi x) + \pi\cos(\pi x)\cos(2\pi x)$$

(F) $f(x) = \log_e(x)\cos(2x)$

$$f'(x) = \log_e x\,(-2\sin 2x) + \cos 2x \cdot \frac{1}{x}$$

$$= -2\sin 2x \log_e x + \frac{\cos(2x)}{x}$$

(G) $y = (x^2+2)^2 e^{-x^2}$

$$\frac{dy}{dx} = (x^2+2)^2 \cdot (-2x)e^{-x^2} + e^{-x^2} \cdot 2(x^2+2) \cdot 2x$$

$$= -2xe^{-x^2}(x^2+2)^2 + 4x(x^2+2)e^{-x^2}$$

(H) $f(x) = \sin^4(x)\cos^3(x)$

$f'(x) = \sin^4 x \cdot 3\cos^2 x \cdot (-\sin x)$

$+ \cos^3 x \cdot 4\sin^3 x \cos x$

$= -3\sin^5 x \cos^2 x$

$+ 4\cos^4 x \sin^3 x$

$= \sin^3 x \cos^2 x \left(4\cos^2 x - 3\sin^2 x\right)$

(Q14) $f(x) = (3x+2)^4 (2x-3)^5$

$f'(x) = (3x+2)^4 \cdot 5(2x-3)^4 \cdot 2$

$+ (2x-3)^5 \cdot 4(3x+2)^3 \cdot 3$

$= 10(3x+2)^4(2x-3)^4 + 12(2x-3)^5(3x+2)^3$

$= 2(3x+2)^3(2x-3)^4\left(5(3x+2) + 6(2x-3)\right)$

$= 2(3x+2)^3(2x-3)^4(27x-8)$

$\therefore f'(x) = 0$

$\Rightarrow x = -2/3, 3/2, 8/27$

(Q15) $y = x^2 e^{3x}$

$\frac{dy}{dx} = x^2 \cdot 3e^{3x} + e^{3x} \cdot 2x$

$= e^{3x}(3x^2 + 2x)$

$= e^{3x} x(3x+2) = 0$

$\therefore x = 0, -2/3$

IF $x=0$, $y=0$

$x = -2/3$, $y = (-2/3)^2 e^{-2} = \frac{4}{9e^2}$

CO-ORD : $(0,0)$, $(-2/3, \frac{4}{9e^2})$

(Q16) $y = (x^2+3x+1)e^x$

$\frac{dy}{dx} = (x^2+3x+1)e^x + (2x+3)e^x$

$= e^x(x^2+5x+4)$

$\frac{dy}{dx} = 0$

$\therefore e^x(x+1)(x+4) = 0$

$\therefore x = -1, -4.$

IF $x=-1$, $y = (1-3+1)e^{-1} = -\frac{1}{e}$

$x=-4$, $y = (16-12+1)e^{-4} = \frac{5}{e^4}$

CO-ORDS : $(-1, -\frac{1}{e})$, $(-4, \frac{5}{e^4})$

(Q17) $f(2) = 4$ $f'(2) = -3$

$g(x) = x^2 f(x)$

$g'(x) = x^2 f'(x) + f(x) \cdot 2x$

$g'(2) = 4f'(2) + f(2) \cdot 4$

$= 4(-3) + 4 \cdot 4$

$= -12 + 16$

$= 4.$

(Q18) (A) $y = \frac{3x}{x+1}$

$\frac{dy}{dx} = \frac{(x+1) \cdot 3 - 3x \cdot 1}{(x+1)^2}$

$= \frac{\cancel{3x}+3-\cancel{3x}}{(x+1)^2}$

$= \frac{3}{(x+1)^2}$

(B) $f(x) = \dfrac{e^x}{x^2-1}$

$$f'(x) = \frac{(x^2-1)e^x - e^x(2x)}{(x^2-1)^2}$$

$$= \frac{e^x(x^2-2x-1)}{(x^2-1)^2}$$

(C) $y = \dfrac{\cos x}{x^2}$

$$\frac{dy}{dx} = \frac{-x^2\sin x - \cos x \cdot 2x}{x^4}$$

$$= \frac{-2x\cos x - x^2\sin x}{x^4}.$$

(D) $f(x) = \dfrac{\log_e 4x + 3}{\tan x}$

$$f'(x) = \frac{\tan x \cdot \frac{1}{x} - (\log_e 4x+3)\sec^2 x}{\tan^2 x}$$

$$= \frac{\tan x - x(\log_e 4x+3)\sec^2 x}{x\tan^2 x}$$

(Q19) $f(3) = 4 \quad f'(3) = -2$

$$g(x) = \frac{x^2}{f(x)}$$

$$g'(x) = \frac{f(x)\cdot 2x - x^2 f'(x)}{(f(x))^2}$$

$$g'(3) = \frac{f(3)\cdot 2\cdot 3 - 3^2 f'(3)}{(f(3))^2}$$

$$= \frac{4\cdot 2\cdot 3 - 9\cdot(-2)}{4^2}$$

$$= \frac{24+18}{16}$$

$$= \frac{42}{16} = \frac{21}{8}$$

(Q20) $f(x) = \dfrac{e^x}{x^2+x-1}$

$$f'(x) = \frac{(x^2+x-1)e^x - e^x(2x+1)}{(x^2+x-1)^2}$$

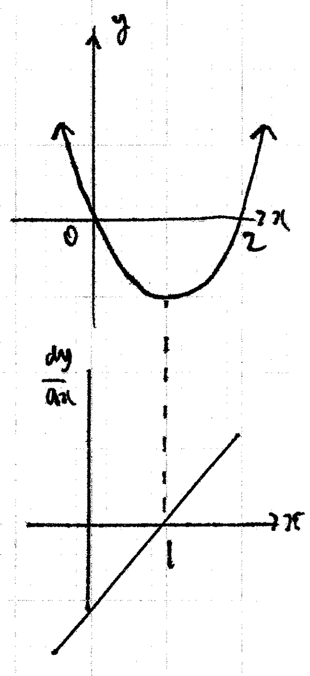
$$= \frac{e^x(x^2-x-2)}{(x^2+x-1)^2}$$

$f'(x) = 0$

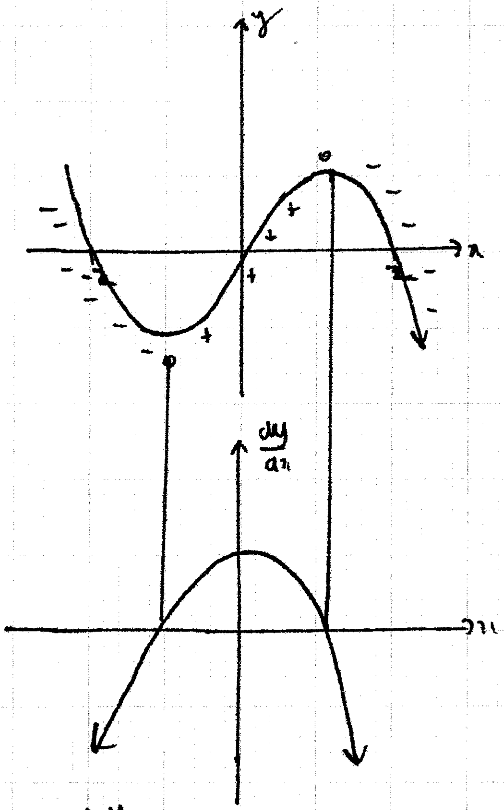
$\therefore (x-2)(x+1) = 0$

$\therefore x = -1, 2$

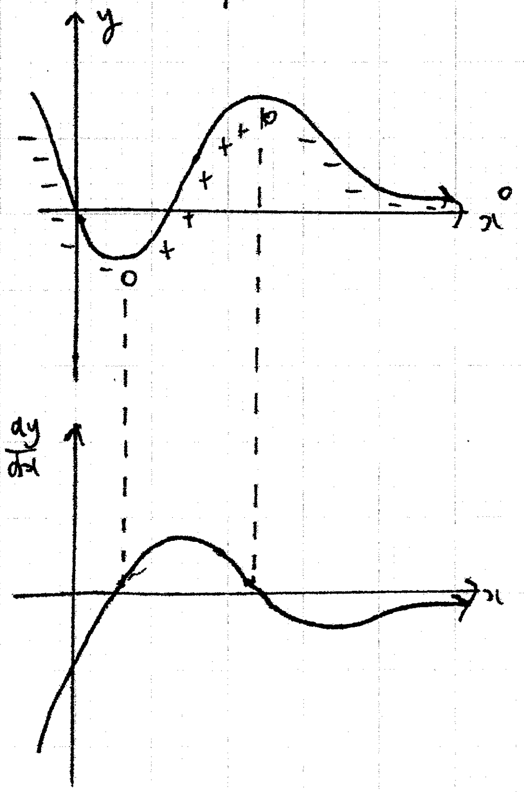
(Q21) (A)



(B)



(C)



Q22 (A) $y = 2(x^2-3x)^{-1}$

$$\frac{dy}{dx} = 2(x^2-3x)^{-2}(2x-3)$$

$$= \frac{2(2x-3)}{(x^2-3x)^2}$$

(B) $f(x) = (\sin x)^{-1}$

$$f'(x) = -(\sin x)^{-2}\cos x$$

$$= -\frac{\cos x}{\sin^2 x}$$

(C) $y = (x^{1/5} + \log_e x)^{-1}$

$$\frac{dy}{dx} = -(x^{1/5} + \log_e x)^{-2} \cdot \left(\tfrac{1}{5}x^{-4/5} + \tfrac{1}{x}\right)$$

$$= -\frac{\left(\frac{1}{5x^{4/5}} + \frac{1}{x}\right)}{(x^{1/5} + \log_e x)^2}$$

(Q23) $f(x) = (x^2+a)e^x$

$$f'(x) = (x^2+a)e^x + e^x(2x)$$

$$= e^x(x^2+2x+a)$$

REQUIRE : $x^2+2x+a > 0$ FOR ALL x

$$\Delta = 2^2 - 4(1)(a) < 0$$

$$\Leftrightarrow \quad 4-4a < 0$$

$$\Leftrightarrow \quad 4 < 4a$$

$$\Leftrightarrow \quad a > 1$$

(Q24) $f(x) = (ax^2+bx+c)\,e^x$

$f'(x) = (ax^2+bx+c)e^x + e^x(2ax+b)$

$= e^x(\,ax^2 + (b+2a)x + (c+b))$

$\therefore \quad a = 2$

$b + 2a = 7$

$b + 4 = 7 \quad \therefore \; b = 3$

$c + b = 4 \quad \therefore \; c = 1$

$$\begin{cases} a = 2 \\ b = 3 \\ c = 1 \end{cases}$$

(Q25) $f(x) = ax^b$

$f'(x) = abx^{b-1}$

FOR INVERSE, SWAP $x \Leftrightarrow y$

$x = ay^b$

$\frac{x}{a} = y^b$

$\left(\frac{x}{a}\right)^{\frac{1}{b}} = (y^b)^{\frac{1}{b}}$

$y = \left(\frac{1}{a}\right)^{\frac{1}{b}} x^{\frac{1}{b}}$

$\therefore \; f^{-1}(x) = \left(\frac{1}{a}\right)^{\frac{1}{b}} x^{\frac{1}{b}}$

IF $f'(x) = f^{-1}(x)$

$\left(\frac{1}{a}\right)^{\frac{1}{b}} x^{\frac{1}{b}} = abx^{b-1}$

$\therefore \; \left(\frac{1}{a}\right)^{\frac{1}{b}} = ab \quad ①$

AND $\frac{1}{b} = b - 1 \quad ②$

SOLVING ② GIVES

$1 = b^2 - b$

$b^2 - b - 1 = 0 \quad (*)$

$b = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2}$

$= \frac{1 \pm \sqrt{5}}{2}.$

SOLVING ①

$\left(\frac{1}{a}\right)^{1/b} = ab$

$\frac{1}{a} = a^b b^b$

$1 = a^{b+1} b^b$

$a^{b+1} = \frac{1}{b^b} = b^{-b}$

$a = (b^{-b})^{\frac{1}{b+1}}$

$= b^{-\frac{b}{b+1}}$

NOTE : FROM $*$ $\quad b + 1 = b^2$

$\therefore \quad a = b^{-\frac{b}{b^2}}$

$= b^{-\frac{1}{b}}$