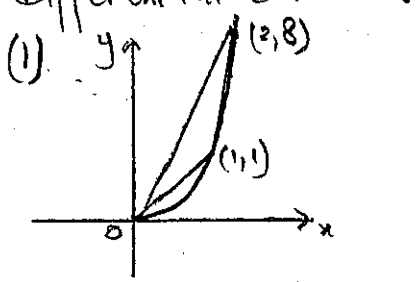
Differential Calculus

(1)



(a) $\frac{\Delta y}{\Delta x} = \frac{1-0}{1-0} = 1$

(b) $\frac{\Delta y}{\Delta x} = \frac{f(2)-f(1)}{2-1} = \frac{2^3-1^3}{1} = \frac{7}{1} = 7$

(c) $\frac{\Delta y}{\Delta x} = \frac{f(2)-f(0)}{2-0} = \frac{8-0}{2} = 4$

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(2) (a) $\frac{\Delta V}{\Delta t} = \frac{1750-1000}{4-2} = \frac{750}{2} = 375$ L/min

(b) $\frac{\Delta V}{\Delta t} = \frac{1750-0}{4-0} = 437.5$ L/min

(c) $\frac{\Delta V}{\Delta t} = \frac{0-1750}{10-4} = \frac{-1750}{6} = 291\frac{2}{3}$ L/min

(d) $\frac{\Delta V}{\Delta t} = \frac{0-0}{10-0} = 0$ L/min

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(3) (a) $\frac{\Delta d}{\Delta t} = \frac{50-22}{8-6} = \frac{28}{2} = 14$ m/s

(b) $\frac{\Delta d}{\Delta t} = \frac{100-50}{10-8} = \frac{50}{2} = 25$ m/s

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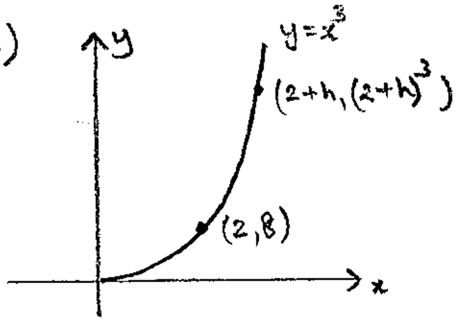
(4)



$$\frac{\Delta y}{\Delta x} = \frac{(2+h)^3-8}{2+h-2}$$
$$= \frac{8+12h+6h^2+h^3-8}{h}$$
$$= \frac{h(12+6h+h^2)}{h}$$
$$= 12+6h+h^2$$

(a) $h=1$, $\frac{\Delta y}{\Delta x} = 12+6+1 = 19$

(b) $h=0.1$, $\frac{\Delta y}{\Delta x} = 12+0.6+0.1^2 = 12.61$

(c) $h=0.01$ $\frac{\Delta y}{\Delta x} = 12+0.06+0.01^2 = 12.0601$

Getting closer to 12 as h decreases

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(5) $f(x) = \frac{1}{x^2}$

$f(1) = 1$

$f(1+h) = \frac{1}{(1+h)^2}$

$$\frac{\Delta y}{\Delta x} = \frac{\frac{1}{(1+h)^2}-1}{1+h-1}$$
$$= \frac{\frac{1-(1+h)^2}{(1+h)^2}}{h}$$
$$= \frac{1-(1+2h+h^2)}{h(1+h)^2}$$
$$= \frac{-2h-h^2}{h(1+h)^2}$$
$$= \frac{-h(2+h)}{h(1+h)^2}$$
$$= -\frac{(2+h)}{(1+h)^2}$$

(a) $h=1$, $\frac{\Delta y}{\Delta x} = \frac{-3}{4}$

(b) $h=0.5$, $\frac{\Delta y}{\Delta x} = \frac{-2.5}{1.5^2} = -1\frac{1}{9}$

(c) $h=0.1$, $\frac{\Delta y}{\Delta x} = \frac{-2.1}{1.1^2} = -1.74$

Getting closer to -2

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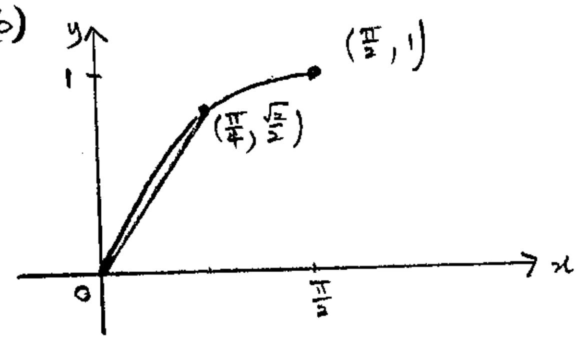
(6)



(a) $\frac{\Delta y}{\Delta x} = \frac{\frac{\sqrt{2}}{2}-0}{\frac{\pi}{4}-0} = \frac{\sqrt{2}}{2} \times \frac{4}{\pi} = \frac{2\sqrt{2}}{\pi}$

(b) $\frac{\Delta y}{\Delta x} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\pi}{2}-\frac{\pi}{4}} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\pi}{4}} = \left(1-\frac{\sqrt{2}}{2}\right)\frac{4}{\pi}$

(c) $\frac{\Delta y}{\Delta x} = \frac{1-0}{\frac{\pi}{2}-0} = \frac{2}{\pi}$

(7) $f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$

$= \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h}$

$= \lim_{h\to 0} \frac{9+6h+h^2-9}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(6+h)}{\cancel{h}}$

$= 6$

(8) $f(x) = \frac{1}{x}$, $x = 1$

$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h}$

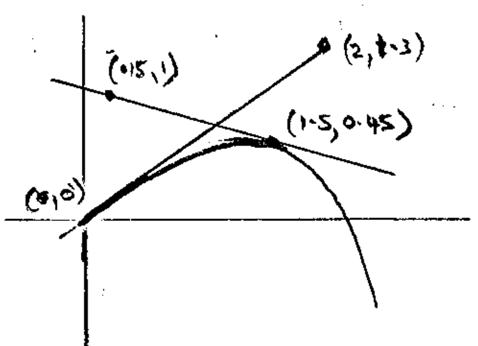
$= \lim_{h\to 0} \frac{\frac{1-(1+h)}{1+h}}{h}$

$= \lim_{h\to 0} \frac{-h}{h(h+1)}$

$= \lim_{h\to 0} -\frac{1}{h+1}$

$= -1$

(9)



$f'(0) \approx \frac{1.3-0}{2-0} = 0.65$

$f'(1.5) \approx \frac{0.45-1}{1.5-0.15} \approx -0.41$

(10) (a) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \frac{3(x+h)-2-(3x-2)}{h}$

$= \frac{3x+3h-2-3x+2}{h}$

$= \frac{3h}{h}$

$= 3$

(b) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h}$

$= \lim_{h\to 0} \frac{3(x^2+2xh+h^2) - 3x^2}{h}$

$= \lim_{h\to 0} \frac{3x^2+6xh+3h^2-3x^2}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(6x+3h)}{\cancel{h}}$

$= \lim_{h\to 0} 6x+3h$

$= 6x$

(c) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{2(x+h)^3 - 2x^3}{h}$

$= \lim_{h\to 0} \frac{2(x^3+3x^2h+3xh^2+h^3) - 2x^3}{h}$

$= \lim_{h\to 0} \frac{\cancel{2x^3}+6x^2h+6xh^2+2h^3-\cancel{2x^3}}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(6x^2+6xh+2h^2)}{\cancel{h}}$

$= \lim_{h\to 0} 6x^2+6xh+2h^2$

$= 6x^2$

(d) $f(x) = 2x^2 - 4x + 3$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{2(x+h)^2 - 4(x+h) + 3 - (2x^2-4x+3)}{h}$

$= \lim_{h\to 0} \frac{\cancel{2x^2}+4xh+2h^2-\cancel{4x}-4h+\cancel{3}-\cancel{2x^2}+\cancel{4x}-\cancel{3}}{h}$

$= \lim_{h\to 0} \frac{h(4x+2h-4)}{h}$

$= \lim_{h\to 0} 4x+2h-4$

$= 4x-4$

(e) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$

$= \lim_{h\to 0} \frac{\frac{2x-2(x+h)}{x(x+h)}}{h}$

$= \lim_{h\to 0} \frac{-2\cancel{h}}{x\cancel{h}(x+h)}$

$= \lim_{h\to 0} \frac{-2}{x(x+h)}$

$= -\frac{2}{x^2}$

10(f) $f(x) = \frac{3}{x^2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{3}{(x+h)^2} - \frac{3}{x^2}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{3x^2 - 3(x+h)^2}{x^2(x+h)^2}}{h}$$
$$= \lim_{h\to 0} \frac{3x^2 - 3x^2 - 6xh - 3h^2}{hx^2(x+h)^2}$$
$$= \lim_{h\to 0} \frac{-h(6x+3h)}{hx^2(x+h)^2}$$
$$= \lim_{h\to 0} \frac{-(6x+3h)}{x^2(x+h)^2}$$
$$= \frac{-6x}{x^2 x^2}$$
$$= -\frac{6}{x^3}$$

11 (a) $f(x) = 2x^3 + 3x^2 - 2x + 4$

$f'(x) = 6x^2 + 6x - 2$

(b) $f(x) = \frac{3}{x} - \frac{2}{x^2}$

$= 3x^{-1} - 2x^{-2}$

$f'(x) = -3x^{-2} + 4x^{-3}$

$= -\frac{3}{x^2} + \frac{4}{x^3}$

(c) $f(x) = \sqrt{x}(3x^2 - x)$

$= x^{1/2}(3x^2 - x)$

$= 3x^{5/2} - x^{3/2}$

$f'(x) = 3 \times \frac{5}{2} x^{3/2} - \frac{3}{2} x^{1/2}$

$= \frac{15x^{3/2}}{2} - \frac{3x^{1/2}}{2}$

$= \frac{15\sqrt{x^3}}{2} - \frac{3\sqrt{x}}{2}$

(d) $g(x) = 3\sqrt{x} + \frac{2}{\sqrt{x}} - 2\sqrt[3]{x}$

$= 3x^{1/2} + 2x^{-1/2} - 2x^{1/3}$

$g'(x) = 3 \times \frac{1}{2} x^{-1/2} - 2 \times \frac{1}{2} x^{-3/2} - 2 \times \frac{1}{3} x^{-2/3}$

$= \frac{3}{2\sqrt{x}} - \frac{1}{\sqrt{x^3}} - \frac{2}{3\sqrt[3]{x^2}}$

(e) $f(x) = (2x-3)^2$

$f(x) = 4x^2 - 12x + 9$

$f'(x) = 8x - 12$

(f) $h(x) = \frac{3x^2 - 3x - 2}{x}$, $x \neq 0$

$= \frac{3x^2}{x} - \frac{3x}{x} - \frac{2}{x}$

$= 3x - 3 - 2x^{-1}$

$h'(x) = 3 + 2x^{-2}$

$= 3 + \frac{2}{x^2}$

(12) $y = 2x^3 - 3x^2 + 2$

$\frac{dy}{dx} = 6x^2 - 6x$

when $x = 2$, $\frac{dy}{dx} = 6(2)^2 - 6(2) = 24 - 12 = 12$

(13) $y = (2x-3)^2$

$= 4x^2 - 12x + 9$

$\frac{dy}{dx} = 8x - 12$

$x = -3$, $\frac{dy}{dx} = -24 - 12 = -36$

(14) $f(x) = ax^3 + bx$

$f'(x) = 3ax^2 + b$

$f'(1) = 0 \Rightarrow 3a + b = 0$ ①

$f'(2) = 9 \Rightarrow 12a + b = 9$ ②

② − ① $\quad 9a = 9$

$a = 1$

① $\quad 3 + b = 0$

$b = -3$

(15) $f(x) = x^3 - 3x^2 + 3$

$f'(x) = 3x^2 - 6x$

Let $3x^2 - 6x = 0$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, 2$

(16) $y = (2x-3)^{2016}$

Let $u = 2x - 3$ $\therefore y = u^{2016}$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= 2016u^{2015} \times 2$

$= 4032(2x-3)^{2015}$

(17) $y = \sqrt{3x-1}$

$y = (3x-1)^{1/2}$

Let $u = 3x-1 \quad \therefore \quad y = u^{1/2}$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= \frac{1}{2}u^{-1/2} \cdot 3$

$= \frac{3}{2\sqrt{u}}$

$= \frac{3}{2\sqrt{3x-1}}$

(18) $y = \frac{1}{2x+5}$

Let $u = 2x+5 \quad \therefore \quad y = \frac{1}{u} = u^{-1}$

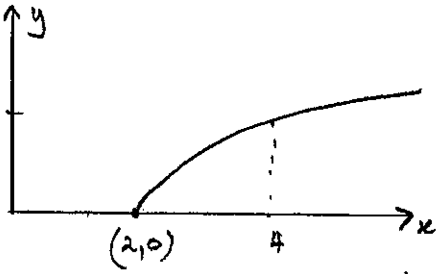
$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= -u^{-2} \times 2$

$= -\frac{2}{u^2}$

$= -\frac{2}{(2x+5)^2}$

(19) $y = \sqrt{2x-4} = \sqrt{2(x-2)}$



Let $u = 2x-4 \quad \therefore y = \sqrt{u} = u^{1/2}$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= \frac{1}{2}u^{-1/2} \cdot 2$

$= \frac{1}{\sqrt{u}}$

$= \frac{1}{\sqrt{2x-4}}$

$x = 10, \frac{dy}{dx} = \frac{1}{\sqrt{16}} = \frac{1}{4}$

20 (a) $f(x) = (3x-2)^4$

$\therefore f'(x) = 4(3x-2)^3 \times 3$

$= 12(3x-2)^3$

(b) $f(x) = \sqrt{2x^2-2x+3}$

$= (2x^2-2x+3)^{1/2}$

$f'(x) = \frac{1}{2}(2x^2-2x+3)^{-1/2} \times (4x-2)$

$= \frac{4x-2}{2\sqrt{2x^2-2x+3}}$

$= \frac{2(x-1)}{2\sqrt{2x^2-2x+3}}$

$= \frac{x-1}{\sqrt{2x^2-2x+3}}$

(c) $g(x) = \frac{1}{3x-1} = (3x-1)^{-1}$

$g'(x) = -(3x-1)^{-2} \times 3$

$= -\frac{3}{(3x-1)^2}$

(d) $f(x) = \frac{1}{(3x-2)^3}$

$= (3x-2)^{-3}$

$\therefore f'(x) = -3(3x-2)^{-4} \times 3$

$= \frac{-9}{(3x-2)^4}$

(e) $h(t) = \sqrt[3]{3t^2-t}$

$= (3t^2-t)^{\frac{1}{3}}$

$h'(t) = \frac{1}{3}(3t^2-t)^{-\frac{2}{3}} \times (6t-1)$

$= \frac{6t-1}{3(3t^2-t)^{2/3}}$

(21) $g(x) = (f(x))^2$

$g'(x) = 2(f(x))f'(x)$

$g'(1) = 2f(1)f'(1)$

$= 2 \times 2 \times 3$

$= 12$

(22) $g(x) = \sqrt{f(x)} = (f(x))^{1/2}$

$g'(x) = \frac{1}{2}(f(x))^{-\frac{1}{2}} \times f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

$g'(2) = \frac{f'(2)}{2\sqrt{f(2)}} = \frac{3}{2\sqrt{4}} = \frac{3}{4}$

23)

(a) $f(x) = \underset{u}{(x^2-2x-1)}\underset{v}{(x^2-3)}$

$f'(x) = uv' + vu'$

$= (x^2-2x-1)(2x) + (x^2-3)(2x-2)$

$= 2x^3 - \cancel{4x^2} - 2x + 2x^3 - 2x^2 - 6x + 6$

$= 4x^3 - 6x^2 - 8x + 6$

(b) $g(t) = (t-1)\sqrt{t+1}$

$= \underset{u}{(t-1)}\underset{v}{(t+1)^{1/2}}$

$g'(t) = uv' + vu'$

$= (t-1)\frac{1}{2}(t+1)^{-\frac{1}{2}} + (t+1)^{\frac{1}{2}}.1$

$= \frac{t-1}{2\sqrt{t+1}} + \sqrt{t+1}$

$= \frac{t-1+t+1}{2\sqrt{t+1}}$

$= \frac{2t}{2\sqrt{t+1}}$

$= \frac{t}{\sqrt{t+1}}$

(c) $y = \underset{u}{(x+2)^3}\underset{v}{(x-2)^3}$

$\frac{dy}{dx} = uv' + vu'$

$= (x+2)^3 3(x-2)^2 + (x-2)^3 . 3(x+2)^2$

$= 3(x+2)^2(x-2)^2[x+2+x-2]$

$= 3(x+2)^2(x-2)^2 . 2x$

$= 6x(x+2)^2(x-2)^2$

(d) $y = (x+1)^5(2x-1)^6$

$\frac{dy}{dx} = uv' + vu'$

$= (x+1)^5\, 6(2x-1)^5 . 2 + (2x-1)^6\, 5(x+1)^4$

$= 12(x+1)^5(2x-1)^5 + 5(2x-1)^6(x+1)^4$

$= (x+1)^4(2x-1)^5[12(x+1) + 5(2x-1)]$

$= (x+1)^4(2x-1)^5(12x+12+10x-5)$

$= (x+1)^4(2x-1)^5(22x+7)$

(24) $y = (x-1)\sqrt{x+1}$

$y = (x-1)(x+1)^{1/2}$

$\frac{dy}{dx} = uv' + vu'$

$= (x-1)\frac{1}{2}(x+1)^{-\frac{1}{2}} + (x+1)^{\frac{1}{2}} \cdot 1$

$= \frac{x-1}{2\sqrt{x+1}} + \sqrt{x+1}$

$= \frac{x-1+x+1}{2\sqrt{x+1}}$

$= \frac{2x}{2\sqrt{x+1}}$

$= \frac{x}{\sqrt{x+1}}$

$x=3,\ \frac{dy}{dx} = \frac{3}{\sqrt{4}} = \frac{3}{2}$

(25) $g(x) = x f(x)$

$g'(x) = uv' + vu'$

$= x f'(x) + f(x)$

$g'(2) = 2 f'(2) + f(2)$

$= 2 \times 3 + 2$

$= 8$

(26) (a) $y = \frac{1}{x^2-1}$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(x^2-1) \times 0 - 1 \times 2x}{(x^2-1)^2}$

$= \frac{-2x}{(x^2-1)^2}$

(b) $f(x) = \frac{x^2}{x-1}$

$f'(x) = \frac{vu' - uv'}{v^2}$

$= \frac{(x-1).2x - x^2 \times 1}{(x-1)^2}$

$= \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2}$

26 (c) $h(t) = \dfrac{\sqrt{t}}{t-4} = \dfrac{t^{\frac{1}{2}}}{t-4}$

$h'(t) = \dfrac{vu' - uv'}{v^2}$

$= \dfrac{(t-4) \times \frac{1}{2} t^{-\frac{1}{2}} - t^{\frac{1}{2}} \times 1}{(t-4)^2}$

$= \dfrac{\dfrac{t-4}{2\sqrt{t}} - \sqrt{t}}{(t-4)^2}$ $\begin{matrix} \times 2\sqrt{t} \\ \times 2\sqrt{t} \end{matrix}$

$= \dfrac{t-4 - \sqrt{t} \times 2\sqrt{t}}{2\sqrt{t}\,(t-4)^2}$

$= \dfrac{t-4-2t}{2\sqrt{t}\,(t-4)^2}$

$= \dfrac{-t-4}{2\sqrt{t}\,(t-4)^2}$

(d) $y = \dfrac{2x+1}{x}$

$= \dfrac{2x}{x} + \dfrac{1}{x}$

$= 2 + \dfrac{1}{x}$

$= 2 + x^{-1}$

$\dfrac{dy}{dx} = -x^{-2}$

$= -\dfrac{1}{x^2}$

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(27) $y = \dfrac{1}{x^2-1}$

$\dfrac{dy}{dx} = \dfrac{-2x}{(x^2-1)^2}$ (from Q26 a)

$x = 3,\ \dfrac{dy}{dx} = \dfrac{-6}{(9-1)^2} = \dfrac{-6}{64} = \dfrac{-3}{32}$

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(28) $y = \dfrac{1}{x+1}$

$\dfrac{dy}{dx} = \dfrac{vu' - uv'}{v^2}$

$= \dfrac{(x+1) \times 0 - 1}{(x+1)^2}$

$= \dfrac{-1}{(x+1)^2}$

Let $-\dfrac{1}{(x+1)^2} = -1$

$\dfrac{1}{(x+1)^2} = 1$

$(x+1)^2 = 1$

$x + 1 = \pm 1$

$x = -1 \pm 1$

$x = -2, 0$

When $x = -2,\ y = \dfrac{1}{-2+1} = -1$ $(-2,-1)$

$x = 0,\ y = \dfrac{1}{1} = 1$ $(0,1)$

Gradient is -1 at $(-2,-1)$ and $(0,1)$

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(29) $y = 3x^2 - 3x + 1$

$\dfrac{dy}{dx} = 6x - 3$

$x = 1,\ \dfrac{dy}{dx} = 6 - 3 = 3$

$x = 1,\ y = 3 - 3 + 1 = 1$ $(1,1)$

$y - y_1 = m(x - x_1)$

$y - 1 = 3(x-1)$

$y = 3x - 2$

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(30) $y = \dfrac{1}{x} = x^{-1}$

$\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

$x = 2,\ \dfrac{dy}{dx} = -\dfrac{1}{4}$

$x = 2,\ y = \dfrac{1}{2}$

$y - y_1 = m(x - x_1)$

$y - \dfrac{1}{2} = -\dfrac{1}{4}(x-2)$

$y = -\dfrac{1}{4}x + \dfrac{1}{2} + \dfrac{1}{2}$

$y = -\dfrac{1}{4}x + 1$

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(31) $y = \sqrt{x+1}$

$y = (x+1)^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \dfrac{1}{2}(x+1)^{-\frac{1}{2}} \times 1 = \dfrac{1}{2\sqrt{x+1}}$

$x = 3,\ \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{3+1}} = \dfrac{1}{4}$

$x = 3,\ y = \sqrt{3+1} = 2$ $(3,2)$

$y - y_1 = m(x - x_1)$

$y - 2 = \dfrac{1}{4}(x-3)$

$y = \dfrac{1}{4}x + \dfrac{5}{4}$

(32)

$x - y = 1$

$\therefore y = x - 1$

$x = 2,\ y = 2 - 1 = 1 \therefore (2,1)$ is point on curve

$m = 1 \therefore x = 2,\ \frac{dy}{dx} = 1$

$y = ax^2 + bx$

Sub in $(2,1)$ $\quad 4a + 2b = 1 \quad (1)$

$\frac{dy}{dx} = 2ax + b$

when $x = 2,\ \frac{dy}{dx} = 1$

$\therefore 4a + b = 1 \quad (2)$

$(1) - (2) \quad b = 0$

$(1) \quad 4a = 1$

$a = \frac{1}{4}$

(33) $y = x^2 - 3x$

$\frac{dy}{dx} = 2x - 3$

$x = 2,\ \frac{dy}{dx} = 4 - 3 = 1$

$\therefore m_N = -1$

$x = 2,\ y = 4 - 6 = -2 \quad (2,-2)$

$y - y_1 = m(x - x_1)$

$y + 2 = -(x - 2)$

$y = -x + 2 - 2$

$y = -x$

(34) $y = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}} \times 1 = \frac{1}{2\sqrt{x+1}}$

$x = 3,\ \frac{dy}{dx} = \frac{1}{2\sqrt{3+1}} = \frac{1}{4}$

$\therefore m_N = -4$

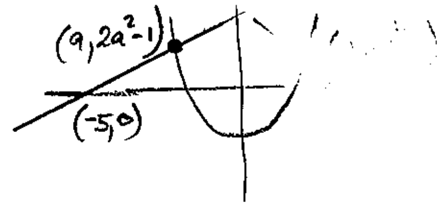
$x = 3,\ y = \sqrt{3+1} = 2 \quad (3,2)$

$y - y_1 = m(x - x_1)$

$y - 2 = -4(x - 3)$

$y = -4x + 14$

(35) $y = 2x^2 - 1$



$\frac{dy}{dx} = 4x$

$x = a,\ \frac{dy}{dx} = 4a$

$\therefore m_N = -\frac{1}{4a}$

Equation of normal

$y - y_1 = m(x - x_1)$

$y - (2a^2 - 1) = -\frac{1}{4a}(x - a)$

when $x = -5,\ y = 0$

$0 - (2a^2 - 1) = -\frac{1}{4a}(-5 - a)$

$2a^2 - 1 = \frac{1}{4a}(-5 - a)$

$8a^3 - 4a = -5 - a$

$8a^3 - 3a + 5 = 0$

$a = -1,\ 8(-1)^3 + 3 + 5 = 0$ true.

equation of normal

$y - (2 - 1) = -\frac{1}{4(-1)}(x - {}^-1)$

$y - 1 = \frac{1}{4}(x + 1)$

$y = \frac{1}{4}x + \frac{5}{4}$

(36) $f(x) = 3x^2 + 2x - 1$

$f'(x) = 6x + 2$

Let $6x + 2 \geq 0 \quad , \quad 6x + 2 \leq 0$

$x \geq -\frac{1}{3} \quad\quad x \leq -\frac{1}{3}$

Increasing on interval $x \geq -\frac{1}{3}$

Decreasing on interval $x \leq -\frac{1}{3}$

(37)(a) $y = x^3 - 8x^2 + 5x + 14$

Y int: $x = 0,\ y = 14 \quad (0,14)$

X int: $y = 0, \quad x^3 - 8x^2 + 5x + 14 = 0$

$y(-1) = (-1)^3 - 8(-1)^2 + 5(-1) + 14$

$= -1 - 8 - 5 + 14$

$= 0$

$\therefore x + 1$ is a factor

$x^3 - 8x^2 + 5x + 14 = (x+1)(x^2 - 9x + 14)$

$= (x+1)(x-2)(x-7)$

X int: $x = -1, 2, 7$

Critical Points

$\frac{dy}{dx} = 3x^2 - 16x + 5$

Let $3x^2 - 16x + 5 = 0$

$$\begin{matrix} 3x & & -1 \\ & \times & \\ x & & -5 \end{matrix}$$

$(3x-1)(x-5) = 0$

$x = \frac{1}{3}, 5$

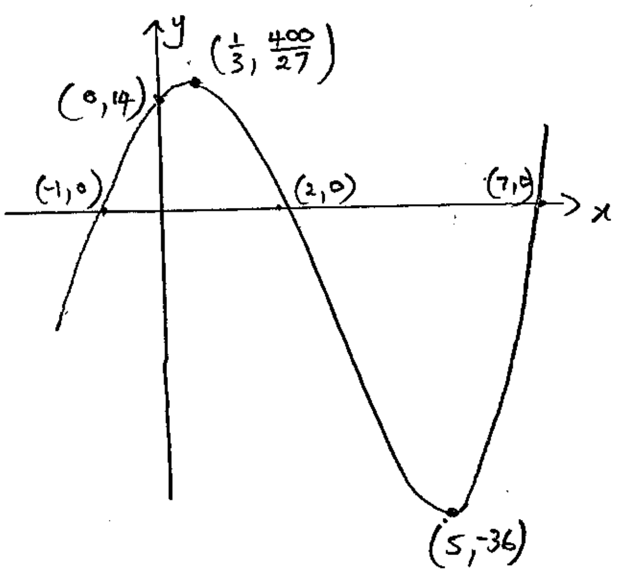
$x = \frac{1}{3}, y = (\frac{1}{3})^3 - 8(\frac{1}{3})^2 + 5(\frac{1}{3}) + 14 = \frac{400}{27}$

$x = 5, y = 5^3 - 8(5)^2 + 5(5) + 14 = -36$

x	0	$\frac{1}{3}$	1	5	6
$\frac{dy}{dx}$	5	0	-8	0	17
	/	—	\	—	/

$(\frac{1}{3}, \frac{400}{27})$ is local max

$(5, -36)$ is local min



(b) $y = 2x^3 - 7x^2 + 9$

Y int : $(0, 9)$

Xint : $y(-1) = 2(-1)^3 - 7(-1)^2 + 9 = 0$

$\therefore x+1$ is factor

$2x^3 - 7x^2 + 9 = (x+1)(2x^2 - 9x + 9)$

$= (x+1)(2x-3)(x-3)$

Xint : $x = -1, \frac{3}{2}, 3$

Critical Points

$\frac{dy}{dx} = 6x^2 - 14x = 0$

$2x(3x-7) = 0$

$x = 0, \frac{7}{3}$

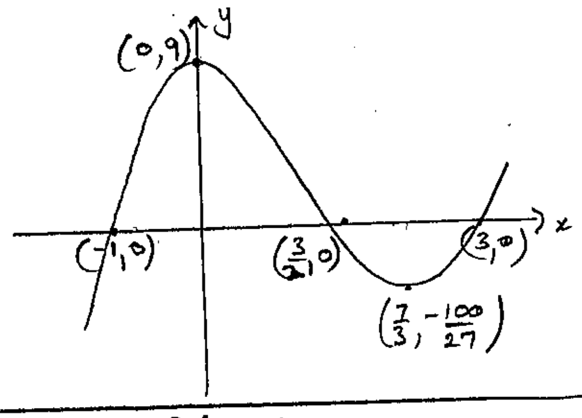
$x = 0, y = 9$

$x = \frac{7}{3}, y = 2(\frac{7}{3})^3 - 7(\frac{7}{3})^2 + 9 = \frac{-100}{27}$

x	-1	0	1	$\frac{7}{3}$	3
$\frac{dy}{dx}$	20	0	-8	0	12
	/	—	\	—	/

$\therefore (0, 9)$ is local max t.p.

$(\frac{7}{3}, -\frac{100}{27})$ is local min t.p.



(c) $f(x) = x^2(x-3)$

Y int : $f(0) = 0$ $(0,0)$

Xint : $x^2(x-3) = 0 \Rightarrow x = 0, 3$

Critical Points

$f(x) = x^3 - 3x^2$

$\therefore f'(x) = 3x^2 - 6x = 0$

$3x(x-2) = 0$

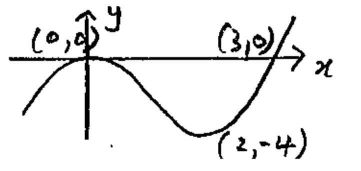
$x = 0, 2$

$x = 0, y = 0$

$x = 2, y = 2^2(2-3) = -4$

x	-1	0	1	2	3
$f'(x)$	9	0	-3	0	9
	/	—	\	—	/

$(0,0)$ is local max and $(2,-4)$ local min turning pt.



37(d) $f(x) = x^3(x-2) = x^4 - 2x^3$

y int: $(0,0)$

X int: $x^3(x-2) = 0$

$x = 0, 2$

Critical pts

$f'(x) = 4x^3 - 6x^2 = 0$

$2x^2(2x-3) = 0$

$x = 0, \frac{3}{2}$

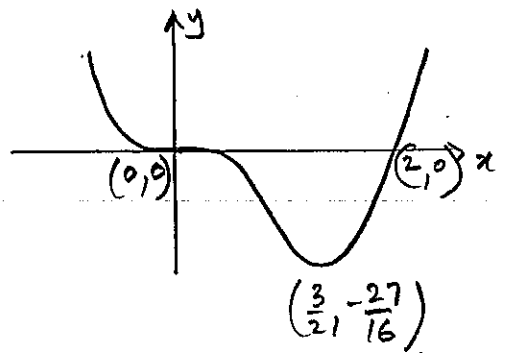
$x = 0, y = 0$

$x = \frac{3}{2}, y = \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 = -\frac{27}{16}$

x	-1	0	1	$\frac{3}{2}$	2
$f'(x)$	-10	0	-2	0	8
	\	—	\	—	/

$(0,0)$ stationary point of inflexion

$\left(\frac{3}{2}, \frac{-27}{16}\right)$ local min turning point



(38) $y = x + \frac{1}{x} = x + x^{-1}$

$\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2}$

Let $1 - \frac{1}{x^2} = 0$

$\frac{1}{x^2} = 1$

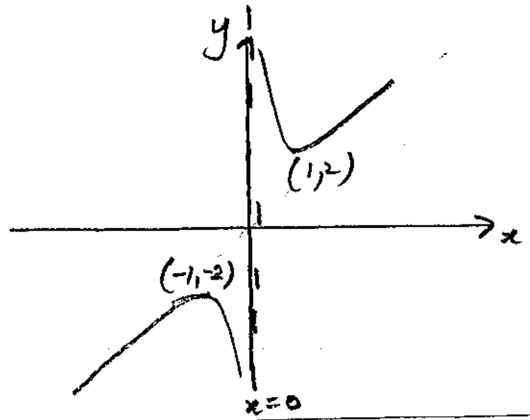
$x^2 = 1$

$x = \pm 1$

$x = 1, y = 1 + 1 = 2 \quad (1,2)$

$x = -1, y = -1 - 1 = -2 \quad (-1,-2)$

$(1,2)$ and $(-1,-2)$ are critical points



(39) $y = x + \frac{1}{x^2} = x + x^{-2}$

$\frac{dy}{dx} = 1 - 2x^{-3} = 1 - \frac{2}{x^3}$

Let $1 - \frac{2}{x^3} = 0$

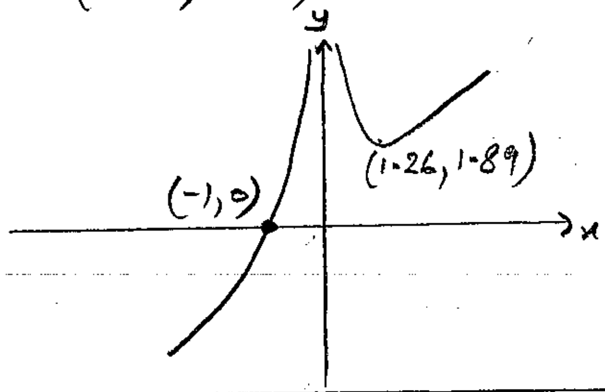
$\frac{2}{x^3} = 1$

$x^3 = 2$

$x = \sqrt[3]{2}$

$x = \sqrt[3]{2}, y = \sqrt[3]{2} + \frac{1}{(\sqrt[3]{2})^2} \simeq 1.89$

$(1.26, 1.89)$



(40) $V(t) = 10t^3(10-t), \quad 0 \le t \le 10$

(a) $V(0) = 0$

$V(2) = 10 \times 2^3 \times 8 = 640$

Av rate change $= \frac{640-0}{2-0} = 320$ L/min

(b) $V(t) = 100t^3 - 10t^4$

$\therefore V'(t) = 300t^2 - 40t^3$

$V'(2) = 300 \times 2^2 - 40 \times 2^3 = 880$

880 L/min

(c) Let $V'(t) = 0$

$\therefore 300t^2 - 40t^3 = 0$

$10t^2(30 - 4t) = 0$

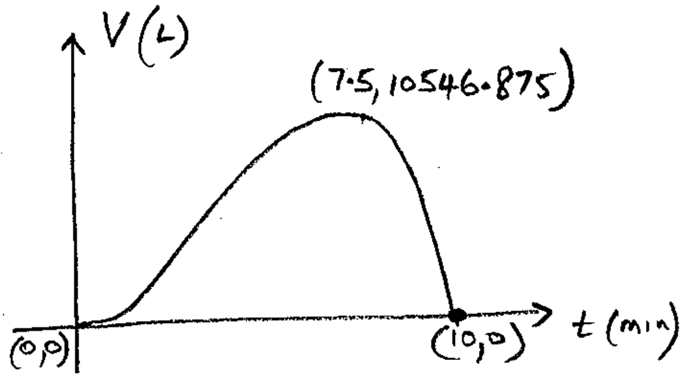
$t = 0, \frac{15}{2}$ ← max

min

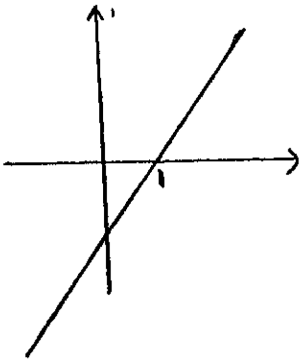
$V\left(\frac{15}{2}\right) = 10\left(\frac{15}{2}\right)^3\left(10-\frac{15}{2}\right) = 10546.875\,L$

Occurs 7.5 min after start

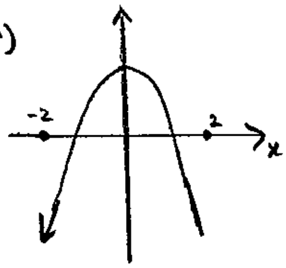
(d)



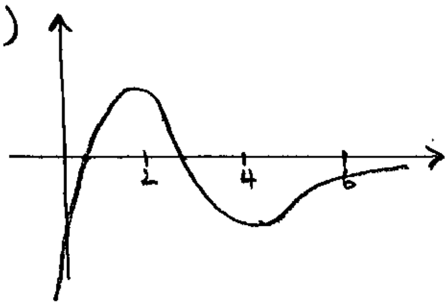
(41) (a)



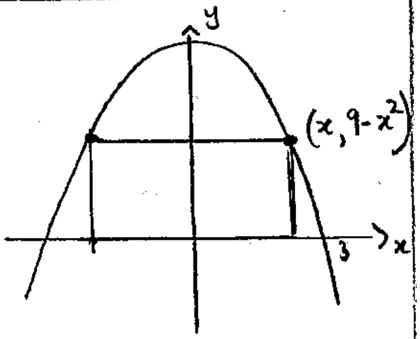
(b)



(c)



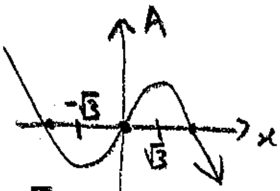
(42)



$A = 2xy$
$= 2x(9-x^2)$
$= 18x - 2x^3$

$A'(x) = 18 - 6x^2$

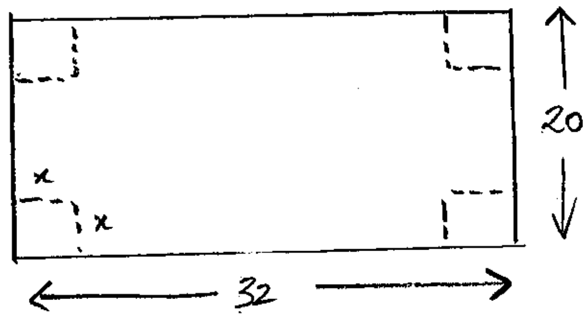
Let $18 - 6x^2 = 0$
$3 - x^2 = 0$
$x = \pm\sqrt{3}$

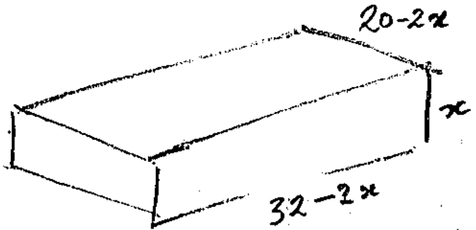


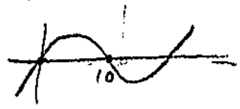
Max occurs when $x = \sqrt{3}$

$A_{max} = 18\sqrt{3} - 2(\sqrt{3})^3$
$= 18\sqrt{3} - 2\times 3\times\sqrt{3}$
$= 12\sqrt{3}$

(43)







$V = x(20-2x)(32-2x)$
$= 4x(10-x)(16-x)$
$= 4x(160 - 10x - 16x + x^2)$
$= 4x(160 - 26x + x^2)$
$= 640x - 104x^2 + 4x^3$

$\frac{dV}{dx} = 640 - 208x + 12x^2$

Let $640 - 208x + 12x^2 = 0$

$160 - 52x + 3x^2 = 0$

$(x-4)(3x-40) = 0$

$\therefore x = 4, \frac{40}{3}$

However $x < 10$ otherwise $20 - 2x < 0$

$\therefore$ $x = 4$ is only solution

considering the shape of the graph we know $x=4$ corresponds to local max. t.p.

Let $x = 4$

$V = 4\times(20-8)(32-8)$
$= 4\times 12\times 24$
$= 1152\ cm^3$

Max Volume $= 1152\ cm^3$

(44)

Aim: Minimize area function

$A = 2\pi r^2 + 2\pi r h$ (1)

Constraint

$V = 2$

$\therefore \pi r^2 h = 2$

$\therefore h = \frac{2}{\pi r^2}$

sub into (1)

$A = 2\pi r^2 + 2\cancel{\pi} r \times \frac{2}{\cancel{\pi} r^2}$

$= 2\pi r^2 + \frac{4}{r}$

$= 2\pi r^2 + 4r^{-1}$

$\therefore \frac{dA}{dr} = 4\pi r - 4r^{-2}$

$= 4\pi r - \frac{4}{r^2}$

Let $\frac{dA}{dr} = 0$

$\therefore 4\pi r - \frac{4}{r^2} = 0$

$4\pi r^3 - 4 = 0$

$r^3 = \frac{1}{\pi}$

$\therefore r = \sqrt[3]{\frac{1}{\pi}}$ $(\simeq 0.7)$

To determine nature of stationary point consider gradient chart

r	$\frac{1}{2}$	$\sqrt[3]{\frac{1}{\pi}}$	1
$\frac{dA}{dr}$	<0	0	>0
	\	—	/

$r = \frac{1}{2}$, $\frac{dA}{dr} = 4\pi(\frac{1}{2}) - \frac{4}{(\frac{1}{2})^2} = 2\pi - 16 < 0$

$r = 1$, $\frac{dA}{dr} = 4\pi - 4 > 0$

$\therefore$ A is minimum when $r = \sqrt[3]{\frac{1}{\pi}}$

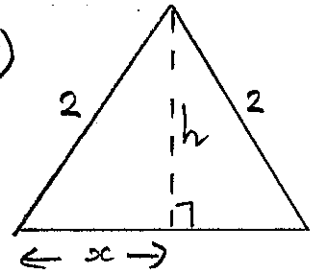
Let $r = \sqrt[3]{\frac{1}{\pi}} = \left(\frac{1}{\pi}\right)^{1/3}$, $A = 2\pi\left[\left(\frac{1}{\pi}\right)^{1/3}\right]^2 + \frac{4}{\left(\frac{1}{\pi}\right)^{1/3}}$

$= \frac{2\pi}{\pi^{2/3}} + 4\pi^{1/3}$

$= 2\pi^{1/3} + 4\pi^{1/3}$

$= 6\sqrt[3]{\pi}$ (Smallest surface area)

(45)



Constraint

$x^2 + h^2 = 4$

$\therefore h = \sqrt{4 - x^2}$

Aim: Maximise area function

$A = \frac{1}{2} bh$

$= \frac{1}{2} \times 2x \times \sqrt{4 - x^2}$

$= x(4 - x^2)^{1/2}$

$\frac{dA}{dx} = uv' + vu'$

$= x \times \frac{1}{\cancel{2}}(4 - x^2)^{-\frac{1}{2}} \times -\cancel{2}x + (4 - x^2)^{\frac{1}{2}} \cdot 1$

$= \frac{-x^2}{\sqrt{4 - x^2}} + \sqrt{4 - x^2}$

$= \frac{-x^2 + \left(\sqrt{4 - x^2}\right)^2}{\sqrt{4 - x^2}}$

$= \frac{-x^2 + 4 - x^2}{\sqrt{4 - x^2}}$

$= \frac{4 - 2x^2}{\sqrt{4 - x^2}}$

Let $\frac{dA}{dx} = 0 \Rightarrow \frac{4 - 2x^2}{\sqrt{4 - x^2}} = 0$

$\therefore 4 - 2x^2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

$x = \sqrt{2}$ $(x > 0)$

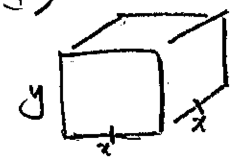
x	1	$\sqrt{2}$	1.5
$\frac{dA}{dx}$	$\frac{2}{\sqrt{3}}$	0	$\frac{-0.5}{\sqrt{1.75}}$
	/	—	\

$\therefore$ A is maximum when $x = \sqrt{2}$

$A_{max} = \sqrt{2}\sqrt{(4 - \sqrt{2}^2)} = \sqrt{2}\sqrt{2} = 2$

$A_{max} = \sqrt{2}$

(46)



Constraint

$A = 2x^2 + 4xy = 360$

$\therefore 4xy = 360 - 2x^2$

$\therefore y = \frac{360 - 2x^2}{4x}$

$y = \frac{180 - x^2}{2x}$

Aim: Maximise volume function

$V = x^2 y$

$= x^2 \times \frac{180 - x^2}{2\cancel{x}}$

$= \frac{1}{2} x (180 - x^2)$

$= 90x - \frac{1}{2}x^3$

$\frac{dV}{dx} = 90 - \frac{3}{2}x^2$

Let $\frac{dV}{dx} = 0 \quad \therefore 90 - \frac{3}{2}x^2 = 0$

$\therefore \frac{3}{2}x^2 = 90$

$\therefore x^2 = 60$

$x = \pm\sqrt{60} \quad (\simeq 7.5)$

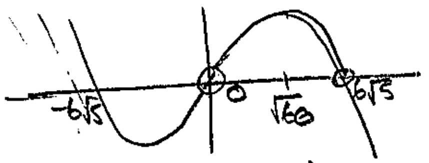
$x = \sqrt{60} \quad (x \geq 0)$

Lets consider shape of graph to determine nature of stationary point at $x = \sqrt{60}$

No $V = \frac{1}{2}x(180 - x^2)$

$= \frac{1}{2}x(\sqrt{180} - x)(\sqrt{180} + x)$

$= \frac{1}{2}x(6\sqrt{5} - x)(6\sqrt{5} + x)$



Clearly V is max when $x = \sqrt{60}$

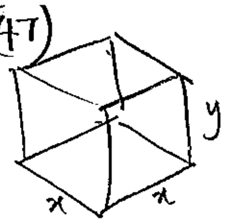
$\therefore V_{max} = 90\sqrt{60} - \frac{1}{2}\sqrt{60}^3$

$= 90\sqrt{60} - 30\sqrt{60}$

$= 60\sqrt{60}$

$= 120\sqrt{15}$

(47)



Constraint

$V = 100$

$x^2 y = 100$

$y = \frac{100}{x^2}$

Aim: Minimise TSA

$A = 2x^2 + 4xy$

$= 2x^2 + 4x \times \frac{100}{x^2}$

$= 2x^2 + \frac{400}{x}$

$= 2x^2 + 400x^{-1}$

$\frac{dA}{dx} = 4x - 400x^{-2}$

$= 4x - \frac{400}{x^2}$

Let $\frac{dA}{dx} = 0 \quad \therefore 4x - \frac{400}{x^2} = 0$

$\therefore \frac{4x^3 - 400}{x^2} = 0$

$4x^3 - 400 = 0$

$x^3 - 100 = 0$

$x^3 = 100$

$x = \sqrt[3]{100} \quad$ (b/w 4 & 5)

Consider gradient chart

x	1	$\sqrt[3]{100}$	5
$\frac{dA}{dx}$	<0	0	>0
	\	—	/

$x = 1, \ \frac{dA}{dx} = \frac{4 - 400}{1} < 0$

$x = 5, \ \frac{dA}{dx} = \frac{4 \times 125 - 400}{25} > 0$

$\therefore$ A is minimum when $x = \sqrt[3]{100}$

Min surface area

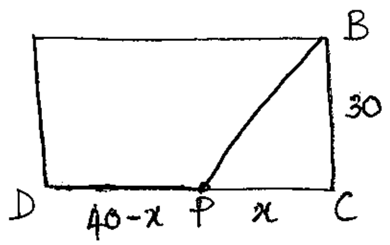
$A = 2\left(100^{1/3}\right)^2 + 400\left(100^{1/3}\right)^{-1}$

$= 2 \times 100^{2/3} + \frac{400}{100^{1/3}}$

$= \frac{2 \times 100 + 400}{\sqrt[3]{100}}$

$= \frac{600}{\sqrt[3]{100}}$

(48)



Let $PC = x \quad \therefore DP = 40 - x$

$PB = \sqrt{x^2+900}$

$T = \text{Time(sand)} + \text{Time(water)}$

$= \dfrac{40-x}{10} + \dfrac{\sqrt{x^2+900}}{2}$

$= \dfrac{40-x}{10} + \dfrac{1}{2}(x^2+900)^{\frac{1}{2}}$

$\dfrac{dT}{dx} = -\dfrac{1}{10} + \dfrac{1}{\cancel{2}} \times \dfrac{1}{2}(x^2+900)^{-\frac{1}{2}} \times \cancel{2}x$

$= -\dfrac{1}{10} + \dfrac{x}{2\sqrt{x^2+900}}$

Let $\dfrac{dT}{dx} = 0$

$\therefore -\dfrac{1}{10} + \dfrac{x}{2\sqrt{x^2+900}} = 0$

$\dfrac{x}{2\sqrt{x^2+900}} = \dfrac{1}{10}$

$10x = 2\sqrt{x^2+900}$

$100x^2 = 4(x^2+900)$

$100x^2 = 4x^2 + 3600$

$96x^2 = 3600$

$x^2 = \dfrac{75}{2}$

$x = \pm\dfrac{\sqrt{75}}{\sqrt{2}}$

$x = \dfrac{\sqrt{150}}{2} \quad (x > 0)$

In this example we will just assume this corresponds to minimum

So min time

$T\left(\dfrac{\sqrt{150}}{2}\right) = \dfrac{40 - \frac{\sqrt{150}}{2}}{10} + \dfrac{\sqrt{\left(\frac{\sqrt{150}}{2}\right)^2 + 900}}{2}$

$= \dfrac{80-\sqrt{150}}{20} + \dfrac{\sqrt{\frac{150}{4}+900}}{2}$

$= \dfrac{80-\sqrt{150}}{20} + \dfrac{\sqrt{3750}}{4}$

$= \dfrac{80-\sqrt{150}}{20} + \dfrac{5\sqrt{25\times150}}{20}$

$= \dfrac{80-\sqrt{150}+25\sqrt{150}}{20}$

$= \dfrac{80+24\sqrt{150}}{20}$

$= \dfrac{80+120\sqrt{6}}{20}$

$= 4 + 6\sqrt{6}$

$\simeq 18.7$ seconds.

(49) $y = ax^2+bx+c$

$\dfrac{dy}{dx} = 2ax+b$

Let $2ax+b=0$

$\therefore x = -\dfrac{b}{2a}$

Graph has turning point at $x = \dfrac{-b}{2a}$

(50) (a) $y = ax^3+bx^2+cx+d$

$\dfrac{dy}{dx} = 3ax^2+2bx+c$

Let $3ax^2+2bx+c=0$

$x = \dfrac{-2b \pm \sqrt{(2b)^2-4(3a)(c)}}{2(3a)}$

$= \dfrac{-2b\pm\sqrt{4b^2-12ac}}{6a}$

$= \dfrac{-2b\pm\sqrt{4(b^2-3ac)}}{6a}$

$= \dfrac{-2b\pm2\sqrt{b^2-3ac}}{6a}$

$= \dfrac{-b\pm\sqrt{b^2-3ac}}{3a}$

So general formula to locate stationary points is

$x = \dfrac{-b\pm\sqrt{b^2-3ac}}{3a}$

(b) (i) $b^2-3ac < 0$

(ii) $b^2-3ac = 0$

(iii) $b^2 - 3ac > 0$

51) Constraint

$x + y = 9$

$\therefore y = 9 - x$

Aim : Maximise $f(x)$

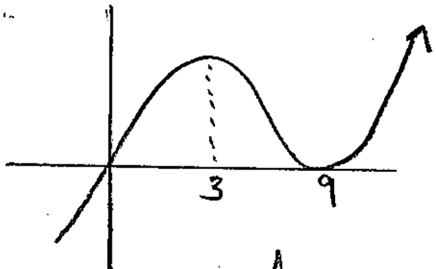
$$f(x) = xy^2$$
$$= x(9-x)^2$$
$$= x(81 - 18x + x^2)$$
$$= 81x - 18x^2 + x^3$$

$\therefore f'(x) = 81 - 36x + 3x^2$

Let $81 - 36x + 3x^2 = 0$

$$27 - 12x + x^2 = 0$$
$$x^2 - 12x + 27 = 0$$
$$(x-9)(x-3) = 0$$
$$x = 9, 3$$

Consider graph of $f(x) = x(9-x)^2$



$\therefore f(x)$ is max when $x = 3$

$x = 3 \Rightarrow y = 9 - 3 = 6$

Two numbers are 3, 6

52)

Constraint :

$\pi r^2 + 2\pi r h = 6$

$\therefore 2\pi r h = 6 - \pi r^2$

$h = \dfrac{6 - \pi r^2}{2\pi r}$

Aim : Maximize Volume

$$V = \pi r^2 h$$
$$= \cancel{\pi} r^{\cancel{2}} \times \frac{6 - \pi r^2}{2\cancel{\pi}\cancel{r}}$$
$$= \frac{1}{2} r (6 - \pi r^2)$$
$$= 3r - \frac{1}{2}\pi r^3$$

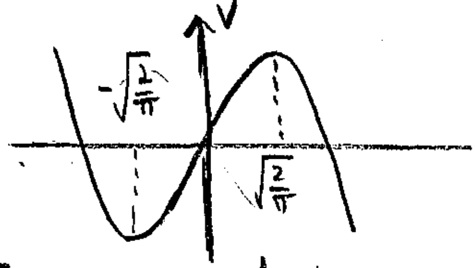
$\dfrac{dV}{dr} = 3 - \dfrac{3}{2}\pi r^2$

Let $3 - \frac{3}{2}\pi r^2 = 0$

$$\frac{3\pi r^2}{2} = 3$$
$$\frac{\pi r^2}{2} = 1$$
$$r^2 = \frac{2}{\pi}$$
$$r = \pm\sqrt{\frac{2}{\pi}}$$
$$r = \sqrt{\frac{2}{\pi}}, \quad (r > 0)$$

Consider shape of graph

$$V = 3r - \frac{1}{2}\pi r^3$$
$$= \frac{1}{2} r (\sqrt{6} - \sqrt{\pi} r)(\sqrt{6} + \sqrt{\pi} r)$$



$r = \sqrt{\frac{2}{\pi}}$ corresponds to maximum turning point

$$\therefore h = \frac{6 - \pi\left(\sqrt{\frac{2}{\pi}}\right)^2}{2\pi\sqrt{\frac{2}{\pi}}}$$
$$= \frac{6 - \cancel{\pi} \times \frac{2}{\cancel{\pi}}}{\sqrt{\frac{8\pi^2}{\pi}}}$$
$$= 4 \times \sqrt{\frac{\cancel{\pi}}{8\pi^{\cancel{2}}}}$$
$$= \sqrt{\frac{16}{8\pi}}$$
$$= \sqrt{\frac{2}{\pi}}$$

(53) Let A be the total number of apples from entire orchard.
Let n = number of additional trees

A = Total # trees × Apples per tree

$$A = (50+n)(800-10n)$$
$$= 40000 - 500n + 800n - 10n^2$$
$$= 40000 + 300n - 10n^2$$

$$\frac{dA}{dn} = 300 - 20n$$

Let $300 - 20n = 0$

$$n = \frac{30\cancel{0}}{2\cancel{0}} = 15$$

This must correspond to maximum turning point as graph of A against n is inverted parabola.

So 15 trees should be added.

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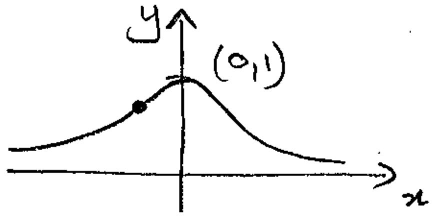
(54) $y = \dfrac{1}{1+x^2}$ $\begin{matrix}\leftarrow u \\ \leftarrow v\end{matrix}$



$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$
$$= \frac{(1+x^2)\times 0 - 1\times 2x}{(1+x^2)^2}$$
$$= \frac{-2x}{(1+x^2)^2}$$

This is the gradient function. Our objective is to find the maximum gradient, ie. to maximize this function

Let $m = \dfrac{-2x}{(1+x^2)^2}$

$$\frac{dm}{dx} = \frac{vu' - uv'}{v^2}$$
$$= \frac{(1+x^2)^2 \times -2 - -2x \times 2(1+x^2) \times 2x}{(1+x^2)^4}$$
$$= \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4}$$
$$= \frac{-2\cancel{(1+x^2)}\left[1+x^2-4x^2\right]}{(1+x^2)^{\cancel{4}\,3}}$$
$$= \frac{-2(1-3x^2)}{(1+x^2)^3}$$

Let $1 - 3x^2 = 0$

$\therefore x^2 = \frac{1}{3}$

$\therefore x = \pm\frac{1}{\sqrt{3}}$

From graph we see slope is positive when $x<0$
So gradient will be max when $x = -\frac{1}{\sqrt{3}}$

Let $x = -\frac{1}{\sqrt{3}}$, $m = \dfrac{-2\times -\frac{1}{\sqrt{3}}}{\left(1+\left(\frac{1}{\sqrt{3}}\right)^2\right)^2}$

$$= \frac{\frac{2}{\sqrt{3}}}{\left(1+\frac{1}{3}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\left(\frac{4}{3}\right)^2} = \frac{2}{\sqrt{3}} \times \frac{9}{16} = \frac{9}{8\sqrt{3}}$$
$$= \frac{9\sqrt{3}}{8\times 3} = \frac{3\sqrt{3}}{8}$$

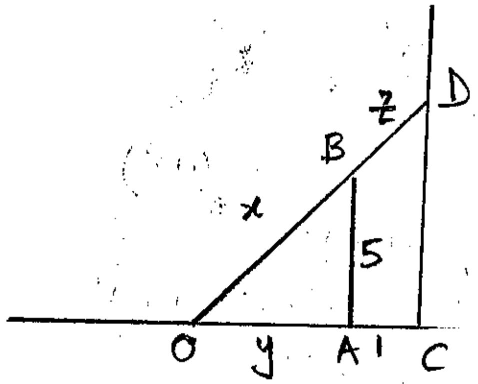
$x = -\frac{1}{\sqrt{3}}$, $y = \dfrac{1}{1+\left(\frac{-1}{\sqrt{3}}\right)^2} = \dfrac{1}{1+\frac{1}{3}} = \frac{3}{4}$

Equation of tangent

$$y - y_1 = m(x - x_1)$$
$$y - \frac{3}{4} = \frac{3\sqrt{3}}{8}\left(x + \frac{1}{\sqrt{3}}\right)$$
$$y = \frac{3\sqrt{3}}{8}x + \frac{3}{8} + \frac{6}{8}$$
$$y = \frac{3\sqrt{3}}{8}x + \frac{9}{8}$$

tangent line with greatest slope

55)



In above diagram OD is ladder, AB is Fence and CD is wall

Using similar triangles we get

$$\frac{x+z}{y+1} = \frac{x}{y}$$

$$\therefore xy + zy = xy + x$$

$$zy = x$$

$$z = \frac{x}{y} \qquad ①$$

Using Pythagoras

$$x^2 = y^2 + 25 \qquad ②$$

Let L = length of ladder

$$\therefore L = x + z$$

$$= x + \frac{x}{y} \qquad ①$$

$$= x + \frac{x}{\sqrt{x^2-25}} \qquad ②$$

$$= x + \frac{x}{(x^2-25)^{1/2}} \qquad \begin{matrix} u \\ v \end{matrix}$$

$$\frac{dL}{dx} = 1 + \frac{vu' - uv'}{v^2}$$

$$= 1 + \frac{(x^2-25)^{\frac{1}{2}}.1 - x\,\frac{1}{2}(x^2-25)^{-\frac{1}{2}}2x}{x^2-25}$$

$$= 1 + \frac{\sqrt{x^2-25} - \frac{x^2}{\sqrt{x^2-25}}}{x^2-25}$$

$$= \frac{x^2-25 + \sqrt{x^2-25} - \frac{x^2}{\sqrt{x^2-25}}}{x^2-25}$$

$$= \frac{(x^2-25)\sqrt{x^2-25} + x^2 - 25 - x^2}{(x^2-25)\sqrt{x^2-25}}$$

$$= \frac{(x^2-25)\sqrt{x^2-25} - 25}{(x^2-25)\sqrt{x^2-25}}$$

Let $\frac{dL}{dx} = 0$

Assume corresponds to minimum turning point

$$\therefore (x^2-25)\sqrt{x^2-25} - 25 = 0$$

$$(x^2-25)\sqrt{x^2-25} = 25$$

Solve on calculator

$$x = \sqrt{5^{4/3} + 25} \qquad (x > 0)$$

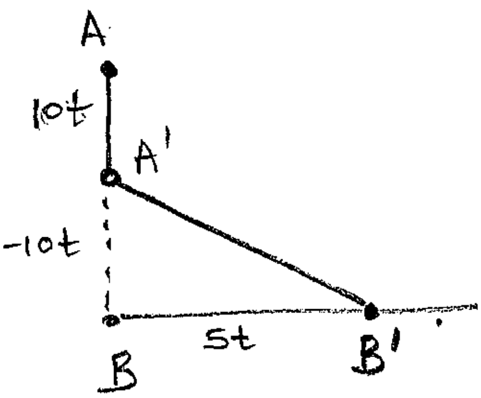
$$\simeq 5.8$$

$\therefore$ Shortest ladder is

$$= 5.8 + \frac{5.8}{\sqrt{5.8^2 - 25}}$$

$$\simeq 7.77$$

(56)



$$A'B' = \sqrt{(3-10t)^2 + 25t^2}$$

$$= \sqrt{9 - 60t + 100t^2 + 25t^2}$$

$$= \sqrt{125t^2 - 60t + 9}$$

Let $D = (125t^2 - 60t + 9)^{\frac{1}{2}}$

$$\frac{dD}{dt} = \frac{1}{2}(125t^2 - 60t + 9)^{-\frac{1}{2}} \times (250t - 60)$$

$$= \frac{125t - 30}{\sqrt{125t^2 - 60t + 9}}$$

Let $\frac{dD}{dt} = 0$

$\therefore 125t - 30 = 0$

$\therefore t = \frac{30}{125}$

D is min when $t = \frac{30}{125}$

$$\text{Min } D = \sqrt{125\left(\frac{30}{125}\right)^2 - 60\left(\frac{30}{125}\right) + 9}$$

$=$

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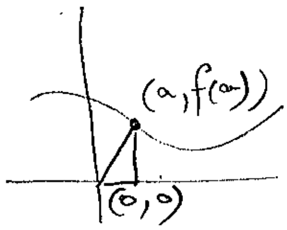
(57)



Let D = distance from $(0,0)$ to $(a, f(a))$

$\therefore D = \sqrt{a^2 + f(a)^2}$ (using Pythag)

$= \left(a^2 + (f(a))^2\right)^{\frac{1}{2}}$

$$\frac{dD}{da} = \frac{1}{2}\left(a^2 + (f(a))^2\right)^{-\frac{1}{2}}(2a + 2f(a)f'(a))$$

$$= \frac{2a + 2f(a)f'(a)}{2\sqrt{a^2 + (f(a))^2}}$$

$$= \frac{a + f(a)f'(a)}{\sqrt{a^2 + (f(a))^2}}$$

To find minimum let $\frac{dD}{da} = 0$

$$\therefore \frac{a + f(a)f'(a)}{\sqrt{a^2 + (f(a))^2}} = 0$$

$\therefore a + f(a)f'(a) = 0$

$\therefore f'(a) = -\frac{a}{f(a)}$

So at the point where distance is minimum $f'(a) = -\frac{a}{f(a)}$

In other words this represents the gradient or slope of the curve at this point.

We will now calculate the gradient of line joining $(0,0)$ to the point $(a, f(a))$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a) - 0}{a - 0} = \frac{f(a)}{a}$$

So $m \times f'(a)$

$= \frac{f(a)}{a} \times -\frac{a}{f(a)}$

$= -1$

Hence if $(a, f(a))$ is the point on the curve which is closest to $(0,0)$ then the curve will be perpendicular to the line joining $(0,0)$ to that point.

(58) $y = \dfrac{1}{f(x)}$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(h)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{f(x) - f(x+h)}{f(x)\, f(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x) - f(x+h)}{h\, f(x) f(x+h)}$$

$$= \lim_{h \to 0} \left(\frac{f(x) - f(x+h)}{h} \times \frac{1}{f(x)} \times \frac{1}{f(x+h)} \right)$$

$$= \lim_{h \to 0} \frac{f(x) - f(x+h)}{h} \times \lim_{h \to 0} \frac{1}{f(x)} \times \lim_{h \to 0} \frac{1}{f(x+h)}$$

$$= \left(- \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right) \times \frac{1}{f(x)} \times \frac{1}{f(x)}$$

$$= - f'(x) \times \frac{1}{(f(x))^2}$$

$$= \frac{-f'(x)}{(f(x))^2}$$ as required.