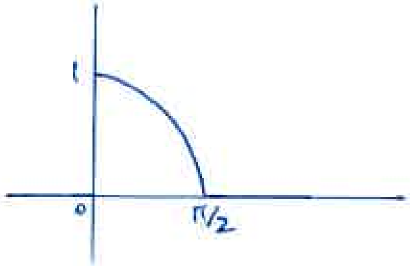
# CONTINUOUS RANDOM VARIABLES

Q1) A) $f(x) = \begin{cases} \cos x & 0 \le x \le \pi/2 \\ 0 & \text{OTHERWISE} \end{cases}$




NOTE: $f(x) \ge 0$ FOR ALL $x \in \mathbb{R}$

AND: $\int_0^{\pi/2} \cos x \, dx$

$= [\sin x]_0^{\pi/2}$

$= \sin \pi/2 - \sin 0$

$= 1$

$\therefore$ f IS A PDF.

B) $Pr(X \le \pi/4)$

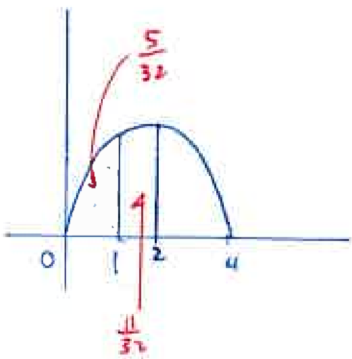
$= \int_0^{\pi/4} \cos x \, dx$

$= [\sin x]_0^{\pi/4}$

$= \sin \pi/4 - \sin 0$

$= \frac{\sqrt{2}}{2}$

Q2) A) $f(x) = \begin{cases} kx(4-x) & 0 \le x \le 4 \\ 0 & \text{OTHERWISE} \end{cases}$




$1 = k \int_0^4 x(4-x) \, dx$

$= k \int_0^4 4x - x^2 \, dx$

$= k \left[ 2x^2 - x^3/3 \right]_0^4$

$= k \left[ (2 \cdot 4^2 - 4^3/3) - (0) \right]$

$= k \cdot \frac{32}{3} \quad \therefore k = \frac{3}{32}$

B) $Pr(X \le 1)$

$= \frac{3}{32} \int_0^1 x(4-x) \, dx$

$= \frac{3}{32} \left[ 2x^2 - x^3/3 \right]_0^1$

$= \frac{\cancel{3}}{32} \cdot \frac{5}{\cancel{3}}$

$= \frac{5}{32}$

C) BY SYMMETRY, $Pr(X \le 2) = \frac{1}{2}$

$\therefore Pr(X \ge 1 \mid X \le 2)$

$= \frac{Pr(X \ge 1 \text{ AND } X \le 2)}{Pr(X \le 2)}$

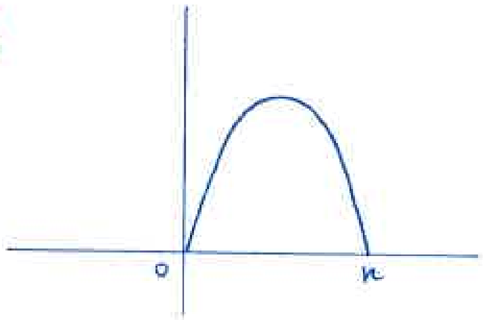
$= \frac{Pr(1 \le X \le 2)}{Pr(X \le 2)}$

$= \frac{11/32}{1/2}$

$= \frac{22}{32}$

$= \frac{11}{16}$

Q3)




$$I = \int_0^n nx - x^2 \, dx$$

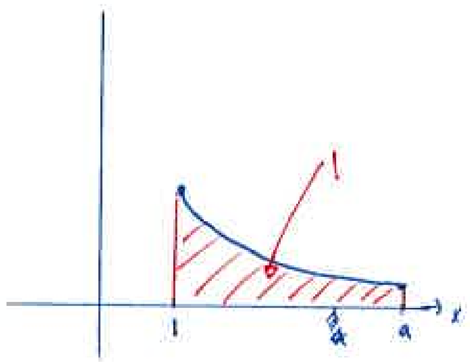
$$= \left[ \frac{nx^2}{2} - \frac{x^3}{3} \right]_0^n$$

$$= \frac{n^3}{2} - \frac{n^3}{3}$$

$$= \frac{n^3}{6}$$

$$\therefore \; n^3 = 6 \Rightarrow \underline{n = \sqrt[3]{6}}$$

Q4)




a)

$$I = \int_1^a \frac{2}{x^2} \, dx$$

$$= \int_1^a 2x^{-2} \, dx$$

$$= \left[ \frac{2x^{-1}}{-1} \right]_1^a$$

$$= \left[ -\frac{2}{x} \right]_1^a$$

$$= -\frac{2}{a} + 2$$

$$\therefore \; \frac{2}{a} = 1 \Rightarrow \underline{a = 2}$$

b)

$$\mu = \int_1^a x f(x) \, dx$$

$$= \int_1^2 x \cdot \frac{2}{x^2} \, dx$$

$$= \int_1^2 \frac{2}{x} \, dx$$

$$= \left[ 2 \log_e x \right]_1^2$$

$$= 2\log_e 2 - 2\log_e 1$$

$$= \underline{\log_e 4}$$

c) $P(1 < X < \log_e 4)$

$$= \int_1^{\log_e 4} \frac{2}{x^2} \, dx$$

$$= \int_1^{\log_e 4} 2x^{-2} \, dx$$

$$= \left[ -\frac{2}{x} \right]_1^{\log_e 4}$$

$$= -\frac{2}{\log_e 4} + 2$$

$$= \frac{2\log_e 4 - 2}{\log_e 4}$$

$$= \frac{\log_e 16 - 2}{\log_e 4} \approx$$

Q5) $f(x) \geq 0$ for all $x \in [0,1)$

$$\int_0^1 4x^3 \, dx = \left[ x^4 \right]_0^1 = 1 - 0 = 1$$

$\therefore$ f is a P.D.F.

$$E(X) = \int_0^1 x f(x) \, dx \qquad E(X^2) = \int_0^1 x^2 f(x) \, dx$$

$$= \int_0^1 4x^4 \, dx \qquad = \int_0^1 4x^5 \, dx$$

$$= \left[ \frac{4x^5}{5} \right]_0^1 \qquad = \left[ \frac{4}{6} x^6 \right]_0^1$$

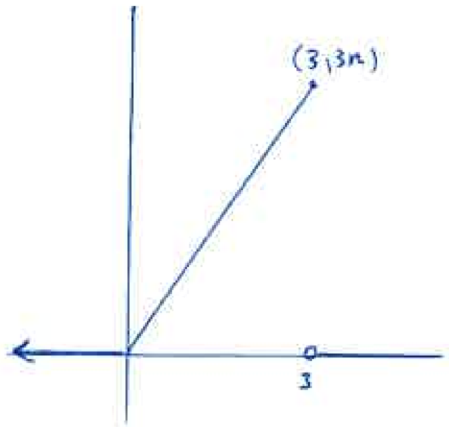
$$= \frac{4}{5} - 0 \qquad = \frac{2}{3} - 0$$

$$= \underline{\frac{4}{5}} \qquad = \underline{\frac{2}{3}}$$

$$SD(X)^2 = E(X^2) - E(X)^2 = \frac{2}{3} - \frac{16}{25} = \frac{2}{75}$$

$$\therefore SD(X) = \sqrt{\frac{2}{75}} = \frac{1}{5}\sqrt{\frac{2}{3}}$$

Q6)




A) $1 = \int_0^3 kx \, dx$

$= \frac{1}{2} \cdot 3 \cdot 3k$

$= \frac{9}{2}k \quad \therefore \quad \underline{k = \frac{2}{9}}$

B) $E(X) = \int_0^3 x f(x) \, dx$

$= \int_0^3 kx^2 \, dx$

$= \left[ \frac{kx^3}{3} \right]_0^3$

$= \left[ \frac{2x^3}{27} \right]_0^3 \qquad (k = \frac{2}{9})$

$= \frac{2}{27} \cdot 3^3 - 0$

$= 2$

c) $E(X^2) = \int_0^3 x^2 f(x) \, dx$

$= \int_0^3 \frac{2}{9} x^3 \, dx$

$= \left[ \frac{2}{36} x^4 \right]_0^3$

$= \frac{81}{18} - 0$

$= \frac{9}{2}$

$SD(X)^2 = E(X^2) - E(X)^2$

$= \frac{9}{2} - 2^2$

$= \frac{1}{2}$

$\therefore \quad SD(X) = \frac{\sqrt{2}}{2}$

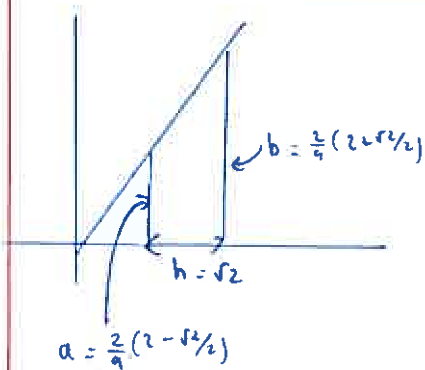
(D) $\mu - \sigma = 2 - \frac{\sqrt{2}}{2}$

$\mu + \sigma = 2 + \frac{\sqrt{2}}{2}$

$P(2 - \frac{\sqrt{2}}{2} < X < 2 + \frac{\sqrt{2}}{2})$

$= \int_{2-\sqrt{2}/2}^{2+\sqrt{2}/2} \frac{2}{9} x \, dx$

AREA OF TRAPEZIUM



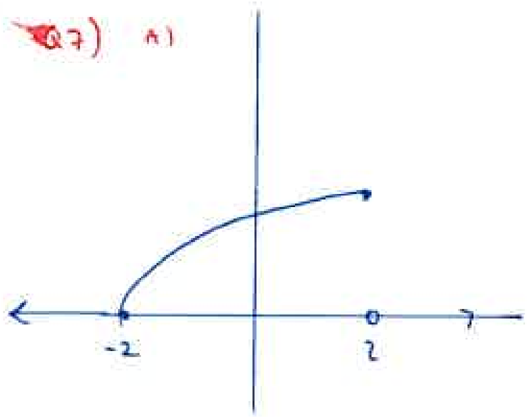

$A = \frac{(a+b)h}{2}$

$= \frac{\frac{2}{9}(2 - \cancel{\sqrt{2}/2} + 2 + \cancel{\sqrt{2}/2})\sqrt{2}}{2}$

$= \frac{\frac{8}{9} \cdot \sqrt{2}}{2}$

$= \frac{4\sqrt{2}}{9}$

Q7) A)




$$I = \int_{-2}^{2} a(x+2)^{1/2}\,dx$$

$$= \left[_2 \frac{a(x+2)^{3/2}}{3/2}\right]_{-2}^{2}$$

$$= a\left[\tfrac{2}{3}(x+2)^{3/2}\right]_{-2}^{2}$$

$$= a\left[\left(\tfrac{2}{3}\cdot 4^{3/2}\right) - 0\right]$$

$$= a \cdot \tfrac{2}{3} \cdot 8$$

$$= a \cdot \tfrac{16}{3} \quad \therefore \quad a = \tfrac{3}{16}$$

B) $E(X) = \int_{-2}^{2} x\, f(x)\, dx$

$$= \int_{-2}^{2} \tfrac{3}{16}\, x\, (x+2)^{1/2}\, dx$$

$$= \underline{\tfrac{2}{5}}$$

C) $E(X^2) = \int_{-2}^{2} \tfrac{3}{16} x^2 (x+2)^{1/2}\, dx$

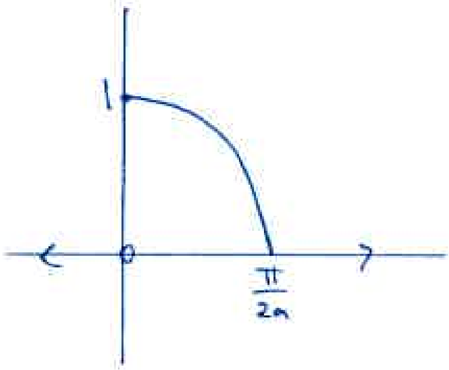
$$= \tfrac{44}{35}$$

$$SD(X) = \left(E(X^2) - E(X)^2\right)^{1/2}$$

$$= \tfrac{8}{5}\sqrt{\tfrac{3}{7}}$$

Q8) $f(x) = \begin{cases} \cos(ax) & x \in [0, \frac{\pi}{2a}] \\ 0 & x \notin [0, \frac{\pi}{2a}] \end{cases}$

NOTE : PERIOD $= T = \frac{2\pi}{a}$.




A) $I = \int_0^{\pi/2a} \cos(ax)\, dx$

$$= \left[\tfrac{1}{a}\sin(ax)\right]_0^{\pi/2a}$$

$$= \tfrac{1}{a}\sin\tfrac{\pi}{2} - \tfrac{1}{a}\cdot 0$$

$$= \tfrac{1}{a} \quad \therefore \quad a = 1$$

B) $E(X) = \int_0^{\pi/2} x\, f(x)\, dx$

$$= \int_0^{\pi/2} x \cos x\, dx$$

$$= \underline{\tfrac{\pi - 2}{2}}$$

C) $E(X^2) = \int_0^{\pi/2} x^2 f(x)\, dx$

$$= \int_0^{\pi/2} x^2 \cos x\, dx$$

$$= \tfrac{\pi^2 - 8}{4}$$

$$SD(X) = \left(E(X^2) - E(X)^2\right)^{1/2}$$

$$= \sqrt{\pi - 3}$$

$$\approx \underline{0.3763}.$$

D) $\Pr(X \leq a) = \frac{1}{2}$

$$\therefore \frac{1}{2} = \int_0^a \cos x \, dx$$
$$= [\sin x]_0^a$$
$$= \sin a - \sin 0$$
$$\therefore \sin a = \frac{1}{2}$$
$$\Rightarrow a = \frac{\pi}{6} \quad (a \in [0, \pi/2])$$

Q9) $1 = \int_0^a x^{1/3} \, dx$

(A)
$$= \left[ \frac{x^{4/3}}{4/3} \right]_0^a$$
$$= \left[ \frac{3}{4} x^{4/3} \right]_0^a$$
$$= \frac{3}{4} a^{4/3}$$
$$\therefore a^{4/3} = 4/3 \Rightarrow a = \left(\frac{4}{3}\right)^{3/4}$$

(B) WANT $E(X) = \int_0^a x f(x) \, dx$
$$= \int_0^a x \cdot x^{1/3} \, dx$$
$$= \int_0^a x^{4/3} \, dx$$
$$= \left[ \frac{x^{7/3}}{7/3} \right]_0^a$$
$$= \left[ \frac{3}{7} x^{7/3} \right]_0^a$$
$$= \frac{3}{7} a^{7/3} \quad \left(a = (4/3)^{3/4}\right)$$
$$= \frac{3}{7} \left(\left(\frac{4}{3}\right)^{3/4}\right)^{7/3}$$
$$= \frac{3}{7} \left(\frac{4}{3}\right)^{7/4}$$
$$\approx 0.7090$$

MEDIAN :
$$\frac{1}{2} = \int_0^a f(x) \, dx$$
$$= \int_0^a \sqrt[3]{x} \, dx$$
$$= \int_0^a x^{1/3} \, dx$$
$$= \left[ \frac{3}{4} x^{4/3} \right]_0^a$$
$$\therefore \frac{1}{2} = \frac{3}{4} a^{4/3}$$
$$\frac{2}{3} = a^{4/3}$$
$$\therefore a = \left(\frac{2}{3}\right)^{3/4} \approx 0.7378$$

Q10) $\int_0^1 x^3 \, dx + \int_1^{7/4} 1 \, dx$

(A)
$$= \left[ \frac{x^4}{4} \right]_0^1 + [x]_1^{7/4}$$
$$= \frac{1}{4} + \left(\frac{7}{4} - 1\right)$$
$$= 1 \qquad \text{ALSO } f(x) \geq 0$$
$\therefore f$ IS A PDF.

(B)
$$\int_0^1 x^3 \, dx + \int_1^a 1 \, dx = 0.95$$
$$0.25 + a - 1 = 0.95$$
$$a - 1 = 0.7$$
$$a = 1.7$$

Q11) A) $1 = \int_0^1 a e^{-t} \, dt$
$$= [-a e^{-t}]_0^1$$
$$= -a[e^{-1} - 1]$$
$$= a(1 - e^{-1})$$
$$\therefore a = \frac{1}{1 - e^{-1}} = \frac{e}{e-1}$$

B) $E(X) = \int_0^1 x \cancel{f(x)}\, dx$

$= \int_0^1 x \cdot a e^{-t}\, dx$

$= a \cdot \frac{e-2}{e}$

$= \frac{\cancel{e}}{e-1} \cdot \frac{e-2}{\cancel{e}}$

$= \frac{e-2}{e-1} \approx 0.4180$ hrs

(C) $Pr(X \geq 0.25) = \int_{1/4}^{1} a e^{-t}\, dt$

15 min $= 0.25$ hrs

$= \frac{e^{3/4}-1}{e-1}$ (EXACT)

$\approx 0.6501$

$Pr($ X $\geq 0.25$ FOUR TIMES $)$

$= 0.6501^4$

$\approx 0.1786$

(D) $Pr(X \leq 0.75) = \int_0^{0.75} a e^{-t}\, dt$

45 min $= 0.75$ hrs

$\approx 0.8347 < 0.9$

$\therefore$ POLICY IS NOT BEING MET.

(E) $\int_0^b a e^{-t}\, dt = 0.9$

$b = \frac{\log_e(\cancel{e})}{\log_e(a+e)} \approx 0.8414$ hrs

~~THEN~~ $= 50$ min

(E) MEDIAN:

$\frac{1}{2} = \int_0^a \alpha e^{-t}\, dx$

$= \frac{e}{e-1} \left[ -e^{-t} \right]_0^a$

$= \frac{e}{e-1} \left[ -e^{-a} + 1 \right]$

$\therefore \frac{e-1}{2e} = 1 - e^{-a}$

$e^{-a} = 1 - \frac{e-1}{2e}$

$= \frac{2e-e+1}{2e}$

$= \frac{e+1}{2e}$

$-a = \log_e\left(\frac{e+1}{2e}\right)$

$a = -\log_e\left(\frac{e+1}{2e}\right) = \log_e\left(\frac{2e}{e+1}\right)$

$\therefore a \approx 0.3799$ hrs

Q12) $1 = \int_0^a 4e^{-2x}\, dx$

A) $1 = \left[ -2e^{-2x} \right]_0^a$

$1 = -2e^{-2a} + 2$

$2e^{-2a} = 1$

$e^{-2a} = \frac{1}{2}$

$-2a = \log_e\left(\frac{1}{2}\right)$

$a = -\frac{\log_e\left(\frac{1}{2}\right)}{2} = \frac{\log_e(2)}{2}.$

(B)

$$\frac{d}{dx} x e^{-2x}$$

$$= x \frac{d}{dx} e^{-2x} + e^{-2x} \frac{d}{dx} x$$

$$= x(-2e^{-2x}) + e^{-2x}$$

$$= -2xe^{-2x} + e^{-2x}$$

$$\therefore \frac{d}{dx}(xe^{-2x}) = -2(x e^{-2x}) + e^{-2x}$$

SOLVE FOR THIS

$$2xe^{-2x} = e^{-2x} - \frac{d}{dx} xe^{-2x}$$

$$\therefore 4xe^{-2x} = 2e^{-2x} - 2\frac{d}{dx} xe^{-2x}$$

$$E(X) = \int_0^a x f(x)\, dx$$

$$= \int_0^a 4xe^{-2x}\, dx$$

$$= \int_0^a 2e^{-2x} - 2\frac{d}{dx} xe^{-2x}\, dx$$

$$= \left[ -e^{-2x} - 2xe^{-2x} \right]_0^a$$

$$= [-e^{-2a} - 2ae^{-2a}] - [-1]$$

$$= -e^{-\log_e 2} - 2\left(\frac{\log_e 2}{2}\right)e^{-\log_e 2} + 1$$

$$= -e^{\log_e \frac{1}{2}} - \log_e 2 \cdot e^{\log_e \frac{1}{2}} + 1$$

$$= -\frac{1}{2} - \log_e 2 \cdot \frac{1}{2} + 1$$

$$= \frac{1 - \log_e 2}{2}$$

(Q13)

(A) $\Pr(W < 1) = \int_0^1 \frac{2}{9} w(3-w)\, dw$

$$= \frac{7}{27} \approx 0.2593 \quad \$5$$

$$\Pr(1 \le w \le 2) = \int_1^2 \frac{2}{9} w(3-w)\, dw$$

$$= \frac{13}{27} \approx 0.4815 \quad \$10$$

$$\Pr(w > 2) = \int_2^3 \frac{2}{9} w(3-w)\, dw$$

$$= \frac{7}{27} \approx 0.2593 \quad \$30$$

(B)

$$E(X) = 270\left(\frac{7}{27} \times 5 + \frac{13}{27} \times 10 + \frac{7}{27} \times 30\right)$$

$$= \frac{270}{27}(7 \times 5 + 13 \times 10 + 7 \times 30)$$

$$= \$5850$$

(C) $X \sim B(10, \frac{7}{27})$ $X = \#$ WEIGHING LESS THAN 1 KG

$\Pr(X \le 3)$ ~~≤ 1/3~~ (USE BINOMCDF)

$$= 0.7539.$$

(D) $\Pr(W > 2) = \cancel{\frac{13}{27}} = \frac{7}{27}$

$n = \#$ CAUGHT $\quad$ Pr(AT LEAST ONE EXCEEDING 2KG) $> 0.95$

$$1 - \Pr(\text{ALL} < 2\text{KG}) > 0.95$$

$$1 - \left(\frac{20}{27}\right)^n > 0.95$$

$$\left(\frac{20}{27}\right)^n < 0.05$$

SOLVE FOR $n > 9.99$ i.e. $n \geq 10$

Q14)

$$1 = \int_0^1 ax^3 \, dx$$

A)

$$= \left[ \frac{ax^4}{4} \right]_0^1$$

$$= \frac{a}{4} \qquad \therefore a = 4$$

B)

$$P(X \le c) = b$$

$$\int_0^c 4x^3 \, dx = b$$

$$\left[ \frac{4x^4}{4} \right]_0^c = b$$

$$[x^4]_0^c = b$$

$$c^4 - 0^4 = b$$

$$\therefore \underline{c = \sqrt[4]{b}.}$$

Q15) A)

$$1 = \int_0^1 ae^x \, dx$$

$$= [ae^x]_0^1$$

$$= ae^1 - ae^0$$

$$\therefore 1 = a(e-1)$$

$$\Rightarrow a = \frac{1}{e-1}$$

B)

$$P(X \le c) = b$$

$$\frac{1}{e-1} \int_0^c e^x \, dx = b$$

$$[e^x]_0^c = b(e-1)$$

$$e^c - e^0 = b(e-1)$$

$$e^c = b(e-1) + 1$$

$$\therefore c = \log_e(be - b + 1)$$

Q16)

A)

$$1 = \int_0^b ax \, dx$$

$$= \left[ \frac{ax^2}{2} \right]_0^b$$

$$= \frac{ab^2}{2}$$

$$\therefore 1 = \frac{ab^2}{2} \Rightarrow a = \frac{2}{b^2}$$

B)

$$E(X) = \int_0^b x f(x) \, dx$$

$$= \frac{2}{b^2} \int_0^b x^2 \, dx$$

$$= \frac{2}{b^2} \left[ \frac{x^3}{3} \right]_0^b$$

$$= \frac{2}{b^2} \left( \frac{b^3}{3} - 0 \right)$$

$$= \frac{2b}{3}.$$

C)

$$E(X^2) = \int_0^b x^2 f(x) \, dx$$

$$= \frac{2}{b^2} \int_0^b x^3 \, dx$$

$$= \frac{2}{b^2} \left[ \frac{x^4}{4} \right]_0^b$$

$$= \frac{2}{b^2} \frac{b^4}{4}$$

$$= \frac{b^2}{2}.$$

$$SD(X)^2 = E(X^2) - E(X)^2$$

$$= \frac{b^2}{2} - \left( \frac{2b}{3} \right)^2$$

$$= b^2 \left( \frac{1}{2} - \frac{4}{9} \right)$$

$$= b^2 \left( \frac{9}{18} - \frac{8}{18} \right)$$

$$= \frac{b^2}{18} \qquad \therefore SD(X) = \frac{b}{\sqrt{18}}$$

$$\underline{= \frac{\sqrt{2}\, b}{3}}$$

Q17)

A) $f(x) = \begin{cases} a x^2 & 0 \le x \le 2 \\ 0 & \text{OTHERWISE} \end{cases}$

$$1 = \int_0^2 a x^2 \, dx$$

$$= \left[ \frac{a x^3}{3} \right]_0^2$$

$$= \left[ \frac{a \cdot 8}{3} \right] - 0$$

$$\therefore \quad \underline{a = \frac{3}{8}}$$

B) $$\frac{3}{2} = \frac{\int_0^b f(x) \, dx}{\int_b^2 f(x) \, dx}$$

$$= \frac{\frac{3}{8} \int_0^b x^2 \, dx}{\frac{3}{8} \int_b^2 x^2 \, dx}$$

$$= \frac{\left[ x^3/3 \right]_0^b}{\left[ x^3/3 \right]_b^2}$$

$$= \frac{b^3/3}{8/3 - b^3/3}$$

$$\therefore \quad \frac{3}{2} = \frac{b^3}{8 - b^3}$$

$$\therefore \quad 3(8 - b^3) = 2b^3$$

$$24 - 3b^3 = 2b^3$$

$$24 = 5b^3$$

$$\frac{24}{5} = b^3$$

$$\Rightarrow \quad b = \sqrt[3]{\frac{24}{5}}.$$

18)

$$\operatorname{VAR}(X) = E\left((X - \mu)^2\right)$$

$$= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx$$

$$= \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f(x) \, dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x) \, dx - 2\mu \int_{-\infty}^{\infty} x f(x) + \mu^2 \int_{-\infty}^{\infty} f(x) \, dx$$

$$= E(X^2) - 2\mu E(X) + \mu^2 \cdot 1$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - E(X)^2$$