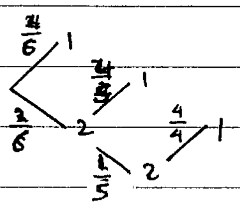
DISCRETE RANDOM VARIABLES + THE BINOMIAL DISTRIBUTION

(Q1)



TOTAL	PROB
1	$\frac{4}{6}$
3	$\frac{2}{6}\cdot\frac{4}{5}$
5	$\frac{2}{6}\cdot\frac{1}{5}\cdot 1$

x	1	3	5
$Pr(X=x)$	$\frac{2}{3}$	$\frac{4}{15}$	$\frac{1}{15}$

$$\text{EXPECTED RETURN} = \frac{2}{3}\cdot 1 + \frac{4}{5}\cdot 3 + \frac{1}{15}\cdot 5$$
$$= \frac{2}{3} + \frac{12}{5} + \frac{1}{3}$$
$$= \frac{10}{15} + \frac{36}{15} + \frac{5}{15}$$
$$= \frac{51}{15}$$
$$= \frac{17}{5}$$
$$= \$3\tfrac{2}{5}$$

(Q2)

x^2	1	0	1	4
x	-1	0	1	2
$Pr(X=x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

$$E(X) = \frac{1}{2}\cdot(-1) + \frac{1}{4}\cdot 0 + \frac{1}{8}\cdot 1 + \frac{1}{8}\cdot 2$$
$$= -\frac{1}{2} + \frac{1}{8} + \frac{1}{4}$$
$$= -\frac{4}{8} + \frac{1}{8} + \frac{2}{8}$$
$$= -\frac{1}{8}$$

$$E(X^2) = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 0 + \frac{1}{8}\cdot 1 + \frac{1}{8}\cdot 4$$
$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{2}$$
$$= \frac{9}{8}$$

$$VAR(X) = E(X^2) - (E(X))^2$$
$$= \frac{9}{8} - \left(\frac{1}{8}\right)^2$$
$$= \frac{72}{64} - \frac{1}{64}$$
$$= \frac{71}{64}$$
$$\therefore SD(X) = \sqrt{\frac{71}{64}} = \frac{\sqrt{71}}{8}$$

(Q3)

+	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

X = SUM

x^2	4	9	16	25	36	49	64
x	2	3	4	5	6	7	8
$Pr(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

$$E(X) = \frac{1}{16}\cdot 2 + \frac{2}{16}\cdot 3 + \frac{3}{16}\cdot 4 + \frac{4}{16}\cdot 5 + \frac{3}{16}\cdot 6 + \frac{2}{16}\cdot 7 + \frac{1}{16}\cdot 8$$
$$= \frac{1}{16}(2 + 6 + 12 + 20 + 18 + 14 + 8)$$
$$= \frac{80}{16}$$
$$= 5$$

$$E(X^2) = \frac{1}{16}(1\cdot 4 + 2\cdot 9 + 3\cdot 16 + 4\cdot 25 + 3\cdot 36 + 2\cdot 49 + 1\cdot 64)$$
$$= \frac{440}{16}$$
$$= 27.5$$

$$VAR(X) = E(X^2) - E(X)^2$$
$$= 27.5 - 5^2$$
$$= 2.5 = 5/2$$

$$SD(X) = \sqrt{5/2}$$

(Q4) $\Sigma p_i = 1$

$3p + p^2 + \frac{3}{16} = 1$

$\times(16) \quad 48p + 16p^2 + 3 = 16$

$16p^2 + 48p - 13 = 0$

$4p \quad -1$
$4p \quad 13$ $\quad (4p-1)(4p+13) = 0$

$\therefore p = 1/4 \quad (\text{AS } p \geq 0)$

x^2	1	4	16
x	1	2	4
$P(X=x)$	$\frac{3}{4}$	$\frac{1}{16}$	$\frac{3}{16}$

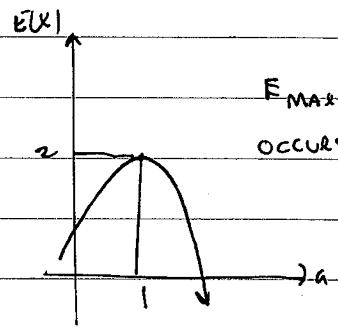
$E(X) = \frac{3}{4}\cdot 1 + \frac{1}{16}\cdot 2 + \frac{3}{16}\cdot 4$
$= \frac{3}{4} + \frac{1}{8} + \frac{3}{4}$
$= \frac{6}{8} + \frac{1}{8} + \frac{6}{8}$
$= \frac{13}{8}$

$E(X^2) = \frac{3}{4}\cdot 1 + \frac{1}{16}\cdot 4 + \frac{3}{16}\cdot 16$
$= \frac{3}{4} + \frac{1}{4} + 3$
$= 4$

$VAR(X) = E(X^2) - E(X)^2$
$= 4 - (13/8)^2$
$= \frac{87}{64}$

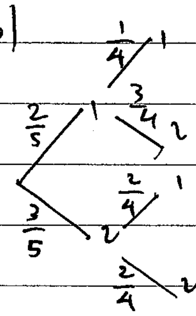
$SD(X) = \sqrt{\frac{87}{64}} = \frac{\sqrt{87}}{8}$

(Q5) $E(X) = \frac{1}{2}(-2a^2) + \frac{1}{3}(3a+6) + \frac{1}{6}\cdot 6$
$= -a^2 + a + 2 + 1$
$= -a^2 + 2a + 1$
$= -(a^2 - 2a - 1)$
$= -(a^2 - 2a + 1 - 1 - 1)$
$= -((a-1)^2 - 2)$
$= -(a-1)^2 + 2$



$E_{MAX} = 2$
OCCURS WHEN $\underline{a = 1}$

(Q6)



		TOTAL
	$\frac{3}{5}\cdot\frac{1}{4} = \frac{1}{10}$	1
	$\frac{2}{5}\cdot\frac{3}{4} = \frac{3}{10}$	3
	$\frac{3}{5}\cdot\frac{2}{4} = \frac{3}{10}$	3
	$\frac{3}{5}\cdot\frac{2}{4} = \frac{3}{10}$	4

x^2	1	9	16
x	1	3	4
$P(X=x)$	$\frac{1}{10}$	$\frac{6}{10}$	$\frac{3}{10}$

$E(X) = \frac{1}{10}\cdot 1 + \frac{6}{10}\cdot 3 + \frac{3}{10}\cdot 4 = \frac{31}{10}$

$E(X^2) = \frac{1}{10}\cdot 1 + \frac{6}{10}\cdot 9 + \frac{3}{10}\cdot 16 = \frac{103}{10}$

$VAR(X) = E(X^2) - E(X)^2 = \frac{69}{100}$

$SD(X) = \frac{\sqrt{69}}{10}$

(Q7) (A) $Pr(BB) = \frac{3}{10} \cdot \frac{3}{10}$
$= \frac{9}{100} = 0.09$

(B) $Pr(\text{LAST BLUE}) = \frac{3}{10} = 0.3$

(C) $Pr(RBRR)$
$= \frac{3}{10} \cdot \frac{3}{10} \cdot \frac{7}{10} \cdot \frac{7}{10}$
$= \frac{441}{10000}$
$= 0.0441$

(D) $Pr(RRRR)$
$= \left(\frac{7}{10}\right)^4$
$= \frac{2401}{10{,}000}$
$= 0.2401$

(E) $Pr(\text{AT LEAST ONE BLUE})$
$= 1 - Pr(\text{ALL RED})$
$= 1 - 0.2401$
$= 0.7599$

(F) $Pr(\text{EXACTLY ONE BLUE})$
$= Pr(BRRR) + Pr(RBRR)$
$+ Pr(RRBR) + Pr(RRRB)$
$= 4 \times \frac{3}{10} \times \frac{7}{10} \times \frac{7}{10} \times \frac{7}{10}$
$= 0.4116$

(G) 2 RED, 2 BLUE :

RRBB BBRR
BRRB RBBR } 6 ARRANGEMENTS
RBRB BRBR

$\therefore Pr(2B, 2R) = 6 \times \left(\frac{3}{10}\right)^2 \times \left(\frac{7}{10}\right)^2$
$= 0.2646$

(H) $Pr(2B \mid \text{AT LEAST ONE BLUE})$

$= \frac{Pr(2B)}{Pr(\text{AT LEAST ONE BLUE})}$

$= \frac{0.2646}{0.7599}$

$= 0.3482$

(Q8) (A) ${}^6C_0 = 1$
(B) ${}^6C_1 = 6$
(C) ${}^6C_2 = 15$
(D) ${}^6C_3 = 20$
(E) ${}^6C_4 = 15$
(F) ${}^6C_5 = 6$
(G) ${}^6C_6 = 1$

(Q9) $X \sim Bi(5, 3/4)$
$Pr(X=2) = {}^5C_2 (3/4)^2 (1/4)^3$
$= 0.0879$

(Q10) X = # OF RED
$X \sim Bi(5, \frac{4}{7})$
$Pr(X=3) = {}^5C_3 \left(\frac{4}{7}\right)^3 \left(\frac{3}{7}\right)^2$
$= 0.3427$

(Q11) X = # OF BULLS-EYES $X \sim Bi(5, \frac{2}{5})$
$Pr(X=2) = {}^5C_2 \left(\frac{2}{5}\right)^2 \left(\frac{3}{5}\right)^3 = 0.3456$

(Q12) X = # OF RED

$X \sim Bi(8, \frac{2}{5})$

$Pr(X=3) = {}^8C_3 (\frac{2}{5})^3 (\frac{3}{5})^5$

$= 0.2787$

(Q13) (A) X = # CORRECT

$X \sim Bi(8, \frac{1}{4})$

$Pr(X=0) = (\frac{3}{4})^8$

$= 0.1001$

(B) $Pr(X=4) = {}^8C_4 (\frac{1}{4})^4 (\frac{3}{4})^4$

$= 0.0865$

(C) $Pr(X \geq 4)$

$= 0.1138$

(D) $Pr(\text{PASS} \mid \text{AT LEAST ONE RIGHT})$

$= \frac{0.1138}{1-0.1001}$

$= 0.1265$

(Q14) (a) $X \sim Bi(8, \frac{1}{2})$ $Pr(X \geq 4) = 0.6367$

(b) $X \sim Bi(8, \frac{1}{3})$ $Pr(X \geq 4) = 0.2586$

(c) $X \sim Bi(8, \frac{1}{4})$ $Pr(X \geq 4) = 0.1138$

(d) $X \sim Bi(8, \frac{1}{5})$ $Pr(X \geq 5) = 0.0563$

(Q15)

(A) $(\frac{1}{6})^2 (\frac{5}{6})^3 = 0.0161$

(B) $X \sim Bi(5, \frac{1}{6})$

$Pr(X=2) = {}^5C_2 (\frac{1}{6})^2 (\frac{5}{6})^3 = 0.1608$

(C) Pr(FIRST 2 SIX | EXACTLY TWO SIX)

$= \frac{\text{Pr(FIRST 2 SIX AND EXACTLY 2 SIX)}}{\text{Pr(EXACTLY TWO SIXES)}}$

$= \frac{\text{Pr(FIRST TWO SIX ONLY)}}{\text{Pr(EXACTLY TWO SIX)}}$

$= \frac{(\cancel{1/6})^2 (\cancel{5/6})^3}{{}^5C_2 (\cancel{1/6})^2 (\cancel{5/6})^3}$

$= 0.1$

(D) $X \sim Bi(5, 1/6)$

$Pr(X \geq 2) = 0.1962.$

(E) $X \sim Bi(5, 1/3)$ ← CAN BE 5 OR 6

$Pr(X=2) = {}^5C_2 (1/3)^2 (2/3)^3$

$= 0.3292$

Q16) $X \sim B(10, \frac{1}{5})$

$E(X) = np = 2$
$P(X \geq E(X)) = P(X \geq 2)$
$= 0.642$

Q17) $X \sim B(n, p)$

$E(X) = np$
$6 = np$

$SD(X) = \sqrt{np(1-p)}$
$2 = \sqrt{np(1-p)}$
$4 = np(1-p)$
$4 = 6(1-p)$
$\frac{2}{3} = 1-p$
$p = \frac{1}{3}$

#6

(Q18) $X \sim Bi(n, \frac{1}{6})$

$$\frac{Pr(X=4)}{Pr(X=5)} = \frac{25}{4}$$

$$\frac{{}^{n}C_4 (\frac{1}{6})^4 (\frac{5}{6})^{n-4}}{{}^{n}C_5 (\frac{1}{6})^5 (\frac{5}{6})^{n-5}} = \frac{25}{4}$$

$$\frac{\frac{\cancel{n!}}{4!(n-4)!} \cancel{\frac{5}{6}}}{\frac{\cancel{n!}}{5!(n-5)!} \cancel{\frac{1}{6}}} = \frac{25}{4}$$

$$\frac{5!(n-5)!\,5}{4!(n-4)!} = \frac{25}{4}$$

$$\cancel{\frac{5 \cdot 5 \cdot 4!(n-5)!}{4!(n-4)(n-5)!} = \frac{25}{4}}$$

$$\frac{1}{n-4} = \frac{1}{4}$$

$$n - 4 = 4$$

$$\therefore \underline{n = 8}$$

(Q19) LET X = # OF MOVES RIGHT

$X \sim Bi(8, \frac{1}{2})$

(A) $Pr(@ \text{ ORIGIN}) = Pr(X=4)$

$$= {}^{8}C_4 (\tfrac{1}{2})^4 (\tfrac{1}{2})^4$$

$$= 0.2734$$

(B) $Pr(\text{LEFT}) = Pr(\text{RIGHT})$

$\therefore Pr(\text{RIGHT}) + Pr(\text{LEFT}) + Pr(\text{ORIGIN}) = 1$

$$2\,Pr(\text{RIGHT}) + Pr(\text{ORIGIN}) = 1$$

$$Pr(\text{RIGHT}) = \frac{1 - Pr(\text{ORIGIN})}{2}$$

$$= \underline{0.3633}$$

(Q20) LET n = # OF ROLLS

$Pr(\text{AT LEAST ONE SIX}) \geq 0.95$

$1 - Pr(\text{NO SIX ROLLED}) \geq 0.95$

$1 - (\frac{5}{6})^n \geq 0.95$

$(\frac{5}{6})^n \leq 0.05$

$\therefore n \geq 16.43$

$n \in \mathbb{N} \therefore n \geq 17$ ROLLS

(Q21) LET n = # OF ROLLS

$Pr(\text{AT LEAST 2 SIXES}) \geq 0.99$

$1 - Pr(\text{0 OR 1 SIX ROLLED}) \geq 0.99$

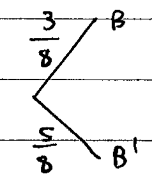
$1 - (\frac{5}{6})^n - {}^{n}C_1 (\frac{1}{6})(\frac{5}{6})^{n-1} \geq 0.99$

$1 - (\frac{5}{6})^n - n(\frac{1}{6})(\frac{5}{6})^{n-1} \geq 0.99$

USE CALC: $n \geq 36.9$

$n \in \mathbb{N} \therefore n \geq 37.$

(Q22)



$Pr(B | X=6)$

$$= \frac{Pr(\text{BIASED AND } X=6)}{Pr(X=6)}$$

$$= \frac{\frac{3}{8}\,{}^{8}C_6 (\frac{3}{4})^6 (\frac{1}{4})^2}{\frac{3}{8}\,{}^{8}C_6 (\frac{3}{4})^6 (\frac{1}{4})^2 + \frac{5}{8}\,{}^{8}C_6 (\frac{1}{2})^6 (\frac{1}{2})^2}$$

$$= \underline{0.6308}$$